

# Path-Integral Quantization of the (2,2) String \*

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## Abstract

A complete treatment of the (2,2) NSR string in flat (2+2) dimensional space-time is given, from the formal path integral over  $N=2$  super Riemann surfaces to the computational recipe for amplitudes at any loop or gauge instanton number. We perform in detail the superconformal gauge fixing, discuss the spectral flow, and analyze the supermoduli space with emphasis on the gauge moduli. Background gauge field configurations in all instanton sectors are constructed. We develop chiral bosonization on punctured higher-genus surfaces in the presence of gauge moduli and instantons. The BRST cohomology is recapitulated, with a new space-time interpretation for picture-changing. We point out two ways of combining left- and right-movers, which lead to different three-point functions.

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\* supported in part by ‘Deutsche Forschungsgemeinschaft’ and ‘Volkswagen-Stiftung’

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# 1 Introduction

String theories with more than one worldsheet supersymmetries have been somewhat neglected in the past, since their critical real space-time dimension of  $4+0$  or  $2+2$  does not promise interesting phenomenology (for a review before 1993, see [1]). This has changed recently, after connections to maximally helicity-violating QCD amplitudes [2, 3] and to F theory [4] were suggested. Since massless kinematics require the choice of  $(2,2)$  signature, our work deals with this case exclusively.

It has been known for some time [5] that open and closed critical  $N=2$  strings each contain each a single massless scalar excitation, which describes self-dual Yang-Mills and self-dual gravity, respectively, in four dimensions. The string reproduces the known fact that there is almost no scattering in these self-dual field theories, and its loop amplitudes should provide guidance for the construction of their proper quantum extensions. Surprisingly, very little was known about  $N=2$  strings beyond the tree-level (see, however, [6]), until Berkovits, Ooguri and Vafa [7, 8, 9] reformulated  $(2,2)$  strings in terms of  $(4,4)$  topological strings,<sup>1</sup> providing an elegant technique for loop calculations. In particular, they demonstrated how the choice of a complex structure in space-time breaks  $N=4$  to  $N=2$  and  $SO(2,2)$  Lorentz symmetry to  $U(1,1)$ .

The traditional  $(2,2)$  NSR formulation should yield the same results in a more elementary fashion. To support this claim, the authors have, together with Ketov, developed the BRST quantization of  $N=2$  strings and its novelties compared to the  $N=1$  case [11, 12, 13]. Recently, we have shown [14] that  $SO(2,2)$  Lorentz invariance can be made manifest in the  $(2,2)$  formulation, when gauge instantons are taken into account and because the string coupling rescales under Lorentz boosts [15].

This work gives a concise and detailed account of the path-integral quantization for the  $(2,2)$  string in flat  $\mathbb{C}^{1,1}$  background, using the NSR formulation. Such a treatment is available for the  $N=1$  string [16, 17], but has been missing in the  $N=2$  case. Although it inevitably contains some material derived and applied in [11, 12, 13], much of it (and certainly most details) cannot be found in the literature, to our knowledge. It is our aim to present a self-consistent basis for subsequent computations of  $N=2$  string amplitudes, and to provide the reader with the necessary tools for such an undertaking.

The paper is organized as follows. After introducing the classical  $(2,2)$  string and its symmetries in Section 2, the gauge-fixing procedure in the path integral is worked out extensively in Section 3. As an interlude, Section 4 covers the superconformal constraint algebra, an operator realization of its spectral flow automorphism, and the BRST symmetry. Section 5 deals with the  $N=2$  supermoduli space, focusing on the  $U(1)$  gauge moduli. In addition, two background gauge field configurations representing topologically non-trivial gauge bundles (nonzero Chern number or instanton sector) are constructed. In Section 6, chiral bosonization in the presence of the gauge moduli is developed, including the evaluation of the sum over bosonic soliton sectors. An instanton-creation operator generalizing spectral flow emerges as a by-product. Physical states and vertex operators are the subject of Section 7, which recaps the BRST cohomology and its

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<sup>1</sup> see also [10] for an earlier note

surprises [18, 11, 19, 14]. The conclusions, Section 8, compare two ways of joining left- and right-movers on the example of the (full) tree-level 3-point function. Three Appendices contain our conventions and give ample details on the spectral-flow operator and on punctured Riemann surfaces.

## 2 The classical (2,2) String

The critical  $N=2$  string lives in  $(2, 2)$  real or  $(1, 1)$  complex space-time dimensions. Let the bosonic string coordinate be denoted by

$$Z^{\pm\mu} = Z^{1\mu} \pm iZ^{2\mu}, \quad \mu = \pm \quad (2.1)$$

where the first superscript  $\pm$  refers to complex conjugation (for real components 1 and 2) and the second superscript  $\mu$  refers to  $(1, 1)$  light-cone coordinates.

The scalar product of two  $(2, 2)$  vectors  $k, p$  can be expressed in this notation by

$$\begin{aligned} k \cdot p &= \frac{1}{2}(k^+ \cdot p^- + k^- \cdot p^+) \\ &= \frac{1}{4}(k^{++}p^{--} + k^{+-}p^{-+} + k^{-+}p^{+-} + k^{--}p^{++}). \end{aligned} \quad (2.2)$$

The dot (on the r.h.s.) is understood as the  $(1, 1)$  scalar product whenever the factors carry one  $\pm$  index. We also introduce

$$\begin{aligned} k^+ \wedge p^+ &= k^{++}p^{+-} - k^{+-}p^{++} \\ k^- \wedge p^- &= k^{-+}p^{--} - k^{--}p^{-+} \end{aligned} \quad (2.3)$$

The same notation applies for the supersymmetric partner  $\Psi$  of  $Z$  which consists of a complex combination of two Majorana spinors,

$$\Psi^{\pm\mu} = \Psi^{1\mu} \pm i\Psi^{2\mu}, \quad (2.4)$$

whose components are labeled by  $\uparrow\downarrow$ -arrows,

$$\Psi^{\pm\mu} = \begin{pmatrix} \Psi_{\uparrow}^{\pm\mu} \\ \Psi_{\downarrow}^{\pm\mu} \end{pmatrix}. \quad (2.5)$$

To construct the  $N=2$  string action one has to couple the  $N=2$  matter to the minimal  $N=2$  supergravity multiplet living on the two-dimensional worldsheet. This supergravity multiplet contains the zweibein  $e_m^a$ , the gravitini  $\chi_m^\pm$  and a  $U(1)$  gauge field  $A_m$ .

The  $N=2$  supersymmetric string action in a flat space-time background is given by [20]

$$S_m = -\frac{1}{4\pi} \int_{\mathcal{M}} d^2z \sqrt{g} \mathcal{L}_m \quad (2.6)$$

with matter Lagrangian <sup>2</sup>

$$\begin{aligned} \mathcal{L}_m &= \frac{1}{2}g^{mn}\partial_m Z^{-\mu}\partial_n Z^+_{\mu} + \frac{i}{2}\bar{\Psi}^{-\mu}\gamma^m \overleftrightarrow{D}_m \Psi^+_{\mu} + A_m \bar{\Psi}^{-\mu}\gamma^m \Psi^+_{\mu} \\ &\quad + (\partial_m Z^{-\mu} + \bar{\Psi}^{-\mu}\chi_m^+) \bar{\chi}_n^- \gamma^m \gamma^n \Psi^+_{\mu} + (\partial_m Z^{+\mu} + \bar{\chi}_m^- \Psi^{+\mu}) \bar{\Psi}^{-\mu} \gamma^n \gamma^m \chi_n^+. \end{aligned} \quad (2.7)$$

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<sup>2</sup> For the definitions of the covariant derivatives  $D_m$  and  $\bar{D}_m$  consult Appendix A.

The complete set of local symmetries [21] acts infinitesimally on the gravity and matter fields as

$$\begin{aligned}
\delta e_m^a &= \xi^n \partial_n e_m^a + e_n^a \partial_m \xi^n + l \varepsilon^a_b e_m^b + 2i(\bar{\chi}_m^- \gamma^a \epsilon^+ - \bar{\epsilon}^- \gamma^a \chi_m^+) + \sigma e_m^a, \\
\delta \chi_m^+ &= \xi^n \partial_n \chi_m^+ + \chi_n^+ \partial_m \xi^n - \frac{1}{2} l \gamma_5 \chi_m^+ + \mathcal{D}_m \epsilon^+ \\
&\quad + \frac{1}{2} \sigma \chi_m^+ + i\alpha \chi_m^+ + i\hat{\alpha} \gamma_5 \chi_m^+ + \gamma_m \eta^+, \\
\delta A_m &= \xi^n \partial_n A_m + A_n \partial_m \xi^n + \varepsilon^{nl} (\bar{\epsilon}^- \gamma_5 \gamma_m \mathcal{D} \chi_m^+ - \mathcal{D}_n \bar{\chi}_l^- \gamma_m \gamma_5 \epsilon^+) \\
&\quad + \partial_m \alpha + \varepsilon_m^n \partial_n \hat{\alpha} + \bar{\chi}_n^- \gamma_m \gamma^n \eta^+ + \bar{\eta}^- \gamma^n \gamma_m \chi_n^+, \\
\delta Z^{+\mu} &= \xi^n \partial_n Z^{+\mu} - 2\bar{\epsilon}^- \Psi^{+\mu}, \\
\delta Z^{-\mu} &= \xi^n \partial_n Z^{-\mu} - 2\bar{\Psi}^{-\mu} \epsilon^+, \\
\delta \Psi^{+\mu} &= \xi^n \partial_n \Psi^{+\mu} - \frac{1}{2} l \gamma_5 \Psi^{+\mu} + i\gamma^n \epsilon^+ (\partial_n Z^{+\mu} + 2(\bar{\chi}_n^- \Psi^{+\mu})) \\
&\quad - \frac{1}{2} \sigma \Psi^{+\mu} + i\alpha \Psi^{+\mu} + i\hat{\alpha} \gamma_5 \Psi^{+\mu},
\end{aligned} \tag{2.8}$$

where the parameters and symmetries are related in the following way

$$\begin{aligned}
\xi &\rightarrow \text{diffeomorphisms,} \\
\epsilon^\pm &\rightarrow N=2 \text{ supersymmetry transformations,} \\
l &\rightarrow \text{Lorentz transformation,} \\
\alpha &\rightarrow U_V(1) \text{ transformation,} \\
\hat{\alpha} &\rightarrow U_A(1) \text{ transformation,} \\
\sigma &\rightarrow \text{bosonic Weyl transformation,} \\
\eta^\pm &\rightarrow \text{fermionic Weyl transformations.}
\end{aligned}$$

In addition, (2.7) is invariant under global  $U(1, 1) \times \mathbb{Z}_2$  target space transformations which act on the  $(1, 1)$  complex indices of the matter fields  $Z$  and  $\Psi$  [5, 11].

For a planar worldsheet and topologically trivial  $U_V(1) \times U_A(1)$  bundle these symmetries suffice to gauge away all gravitational degrees of freedom [21]. After switching to light-cone coordinates and performing the Wick rotation to a Euclidean worldsheet, the resulting gauge-fixed Lagrangian

$$\mathcal{L}_m^{fix} = \partial_z Z^{-\mu} \partial_{\bar{z}} Z^+_{\mu} + \partial_{\bar{z}} Z^{-\mu} \partial_z Z^+_{\mu} + \Psi_{\uparrow}^{-\mu} \overleftrightarrow{\partial}_z \Psi_{\uparrow}^{+\mu} + \Psi_{\downarrow}^{-\mu} \overleftrightarrow{\partial}_{\bar{z}} \Psi_{\downarrow}^{+\mu} \tag{2.9}$$

has an enhanced target space symmetry, namely  $SO(2, 2) \supset U(1, 1)$ . This gauge is globally appropriate only for tree-level and zero-instanton calculations of string amplitudes since beyond that one has to cope with the  $N=2$  moduli, the remnants of the gauge-fixing procedure. Nevertheless, it can always be used locally, as long as one remembers to compute the free field correlators in a non-trivial metric and gauge background.

In the presence of vertex operators (which create infinitesimal holes in the worldsheet) we need to specify the holonomies of the worldsheet fermions around the punctures even on tree level. The global subgroup of  $U_V(1) \times U_A(1)$  acts as

$$\begin{aligned}
\Psi_{\downarrow}^{\pm\mu} &\longrightarrow e^{\pm 2\pi i \rho} \Psi_{\downarrow}^{\pm\mu}, & \chi_{\downarrow m}^{\pm} &\longrightarrow e^{\pm 2\pi i \rho} \chi_{\downarrow m}^{\pm}, \\
\Psi_{\uparrow}^{\pm\mu} &\longrightarrow e^{\pm 2\pi i \bar{\rho}} \Psi_{\uparrow}^{\pm\mu}, & \chi_{\uparrow m}^{\pm} &\longrightarrow e^{\pm 2\pi i \bar{\rho}} \chi_{\uparrow m}^{\pm},
\end{aligned} \tag{2.10}$$

so one can have a different constant pair  $\rho, \bar{\rho}$  for each puncture (and also for each standard homology cycle). Stated differently, the worldsheet fermions are sections of a twisted holomorphic line bundle. But there is more. We shall see that harmonic gauge transformations serve to connect all different twists (for the same topology). The continuous transformation

$$(\rho_\ell, \bar{\rho}_\ell) \longrightarrow (\rho'_\ell, \bar{\rho}'_\ell) \quad (2.11)$$

is called ‘spectral flow’. Hence, no matter what choice of holonomies one starts with, the path integral is going to average over all of them.

### 3 Gauge Fixing and Path-Integral Quantization

The appropriate description for higher-loop string interactions is given by the path integral over higher-genus worldsheet geometries. Since the 2d supergravity multiplet has no dynamical degrees of freedom it can be integrated out almost completely. In order to obtain a well-defined expression for the remaining path integral one has to split the gravitational fields in a gauge part and a non-gauge part, namely the  $N=2$  moduli. The gauge part can of course be expressed by the gauge parameters. This change of variables in the path-integral measure will be accompanied by a Jacobian which can be written as a path integral over ghost and antighost fields. The integral over the gauge parameters is divergent and yields the volume of the symmetry group which is factored out.

For the calculation of the Jacobian it is convenient to formulate the infinitesimal transformation laws of the supergravity multiplet in a covariant way. Let us start with the zweibein,

$$h_b^a := e_b^m \delta e_m^a = D_b \xi^a - 2i(\bar{\epsilon}^- \gamma^a \chi_b^+ - \bar{\chi}_b^- \gamma^a \epsilon^+) - (l + \omega_c \xi^c) \epsilon_b^a + \sigma \delta_b^a - I^a \epsilon_{bc} \xi^c, \quad (3.1)$$

where  $I^a = i\epsilon^{bc} \bar{\chi}_b^- \gamma^a \chi_c^+$ . Let us next consider the transformations for the gravitini and the  $U(1)$  gauge field lifted to the tangent space:

$$\begin{aligned} \delta \chi_a^\pm &= \delta(e_a^m \chi_m^\pm) = -h_a^b \chi_b^\pm + e_a^m \delta \chi_m^\pm \\ &= -h_a^b \chi_b^\pm + D_a \xi^b \chi_b^\pm + \xi^b D_b \chi_a^\pm + \mathcal{D}_a \epsilon^\pm \pm i\alpha \chi_a^\pm \mp i\gamma_5 \hat{\alpha} \chi_a^\pm, \\ &\quad -\frac{1}{2}(l + \omega_c \xi^c) \gamma_5 \chi_a^\pm + \frac{1}{2} \sigma \chi_a^\pm + \gamma_a \eta^\pm - \chi_c^\pm \epsilon_{ab} \xi^b I^c, \\ \delta A_a &= \delta(e_a^m A_m) = -h_a^b A_b + e_a^m \delta A_m \\ &= -h_a^b A_b + D_a \xi^b A_b + \xi^b D_b A_a + \epsilon^{bc} (\bar{\epsilon}^- \gamma_5 \gamma_a \mathcal{D}_b \chi_c^+ - \mathcal{D}_b \bar{\chi}_c^- \gamma_a \gamma_5 \epsilon^+) \\ &\quad - I^d (\bar{\epsilon}^- \gamma_5 \gamma_a \chi_d^+ - \bar{\chi}_d^- \gamma_a \gamma_5 \epsilon^+) \\ &\quad + \partial_a \alpha + \epsilon_a^b \partial_b \hat{\alpha} + \bar{\chi}_b^- \gamma_a \gamma^b \eta^+ + \bar{\eta}^- \gamma^b \gamma_a \chi_b^+. \end{aligned} \quad (3.2)$$

To separate the ranges of the individual transformations it is necessary to redefine some of their parameters,

$$\tilde{\sigma} = \sigma + \frac{1}{2} D_a \xi^a,$$

$$\begin{aligned}\tilde{l} &= l + \omega_c \xi^c + \frac{1}{2}(D_{\bar{z}}\xi^{\bar{z}} - D_z\xi^z), \\ \tilde{\eta}^\pm &= \eta^\pm + \frac{1}{2}\gamma^a \mathcal{D}_a \epsilon^\pm.\end{aligned}\tag{3.3}$$

The first two redefinitions separate the diffeomorphisms, Weyl and Lorentz transformations, whereas the third redefinition separates fermionic Weyl and supersymmetry transformations. We can now eliminate the components  $h_z{}^z$  and  $h_{\bar{z}}{}^{\bar{z}}$  by combinations of Lorentz ( $l$ ) and Weyl ( $\sigma$ ) transformations and thus fix these two symmetries.

Using the inhomogenous fermionic Weyl transformations ( $\eta_{\uparrow\downarrow}^\pm$ ), it is also possible to gauge away the components  $\chi_{\bar{z}\downarrow}^\pm$  and  $\chi_{z\uparrow}^\pm$  of the gravitini. Thus we will only have to introduce ghosts for the diffeomorphisms, supersymmetry and the  $U_V(1), U_A(1)$  transformations. In order to separate holomorphically the range of the remaining unfixed symmetries (see [22]), one has to redefine the supersymmetry and  $U(1)$  gauge parameters as follows,

$$\begin{aligned}\tilde{\epsilon}_\uparrow^\pm &= \epsilon_\uparrow^\pm + \xi^{\bar{z}}\chi_{\bar{z}\uparrow}^\pm, \\ \tilde{\epsilon}_\downarrow^\pm &= \epsilon_\downarrow^\pm + \xi^z\chi_{z\downarrow}^\pm, \\ \tilde{\alpha} - \tilde{\tilde{\alpha}} &= \alpha - \hat{\alpha} + \xi^{\bar{z}}A_{\bar{z}}, \\ \tilde{\alpha} + \tilde{\tilde{\alpha}} &= \alpha + \hat{\alpha} + \xi^zA_z.\end{aligned}\tag{3.4}$$

The final result for the covariant transformations of the zweibein, the gravitini, and the gauge connection is given by

$$\begin{aligned}h_{\bar{z}}{}^z &= D_{\bar{z}}\xi^z - 4(\tilde{\epsilon}_\uparrow^-\chi_{\bar{z}\uparrow}^+ - \chi_{\bar{z}\uparrow}^-\tilde{\epsilon}_\uparrow^+), \\ h_z{}^{\bar{z}} &= D_z\xi^{\bar{z}} + 4(\tilde{\epsilon}_\downarrow^-\chi_{z\downarrow}^+ - \chi_{z\downarrow}^-\tilde{\epsilon}_\downarrow^+), \\ \delta\chi_{\bar{z}\uparrow}^\pm &= \xi^z D_z\chi_{\bar{z}\uparrow}^\pm - \frac{1}{2}D_z\xi^z\chi_{\bar{z}\uparrow}^\pm + \mathcal{D}_{\bar{z}}\tilde{\epsilon}_\uparrow^\pm \pm i(\tilde{\alpha} - \tilde{\tilde{\alpha}})\chi_{\bar{z}\uparrow}^\pm, \\ \delta\chi_{z\downarrow}^\pm &= \xi^{\bar{z}} D_{\bar{z}}\chi_{z\downarrow}^\pm - \frac{1}{2}D_{\bar{z}}\xi^{\bar{z}}\chi_{z\downarrow}^\pm + \mathcal{D}_z\tilde{\epsilon}_\downarrow^\pm \pm i(\tilde{\alpha} + \tilde{\tilde{\alpha}})\chi_{z\downarrow}^\pm, \\ \delta A_{\bar{z}} &= \xi^z D_z A_{\bar{z}} - 4iD_z\xi^{\bar{z}}\chi_{\bar{z}\uparrow}^-\chi_{\bar{z}\uparrow}^+ \\ &\quad + 2i[D_z\chi_{\bar{z}\uparrow}^+\tilde{\epsilon}_\uparrow^- - D_z\chi_{\bar{z}\uparrow}^-\tilde{\epsilon}_\uparrow^+ + \chi_{\bar{z}\uparrow}^-D_z\tilde{\epsilon}_\uparrow^+ - \chi_{\bar{z}\uparrow}^+D_z\tilde{\epsilon}_\uparrow^-] + \partial_{\bar{z}}(\tilde{\alpha} - \tilde{\tilde{\alpha}}), \\ \delta A_z &= \xi^{\bar{z}} D_{\bar{z}} A_z - 4iD_{\bar{z}}\xi^z\chi_{z\downarrow}^-\chi_{z\downarrow}^+ \\ &\quad + 2i[D_{\bar{z}}\chi_{z\downarrow}^+\tilde{\epsilon}_\downarrow^- - D_{\bar{z}}\chi_{z\downarrow}^-\tilde{\epsilon}_\downarrow^+ + \chi_{z\downarrow}^-D_{\bar{z}}\tilde{\epsilon}_\downarrow^+ - \chi_{z\downarrow}^+D_{\bar{z}}\tilde{\epsilon}_\downarrow^-] + \partial_z(\tilde{\alpha} + \tilde{\tilde{\alpha}}).\end{aligned}\tag{3.5}$$

It factorizes holomorphically.<sup>3</sup> In particular, the linear combinations  $U_V(1) \pm U_A(1)$  act independently on left- and right-movers. We are now able to construct the ghost action directly from the transformation laws (3.5).

For the remainder of this section, we will concentrate only on the holomorphic half of (3.5), i.e. its first, third, and fifth equation. An arbitrary variation of the gravitational

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<sup>3</sup> The anholomorphic piece of  $\delta A_{\bar{z}}$ , containing  $\chi_{\bar{z}\uparrow}^-\chi_{\bar{z}\uparrow}^+$ , disappears when choosing delta-function support for the gravitini (see Section 5.)

multiplet can be achieved by a gauge transformation (3.5) plus an infinitesimal shift in the  $N=2$  moduli space,

$$\begin{pmatrix} \tilde{h}_{\bar{z}}^z \\ \tilde{\delta}\chi_{\bar{z}\uparrow}^\pm \\ \tilde{\delta}A_{\bar{z}} \end{pmatrix} = \mathcal{P} \begin{pmatrix} \xi^z \\ \tilde{\epsilon}_\uparrow^\pm \\ \tilde{\alpha} - \tilde{\alpha} \end{pmatrix} + \begin{pmatrix} \mu_{i\bar{z}}^z & 0 & 0 \\ 0 & \nu_j^\pm & 0 \\ 0 & 0 & \omega_{k\bar{z}} \end{pmatrix} \begin{pmatrix} t^i \\ \zeta_{\bar{z}\uparrow}^{j\pm} \\ C^k \end{pmatrix} . \quad (3.6)$$

The operator  $\mathcal{P}$  in (3.6) is read off the holomorphic half of (3.5) as

$$\mathcal{P} = \begin{pmatrix} D_{\bar{z}} & 4\chi_{\bar{z}\uparrow}^- & 4\chi_{\bar{z}\uparrow}^+ & 0 \\ D_z\chi_{\bar{z}\uparrow}^+ - \frac{1}{2}\chi_{\bar{z}\uparrow}^+ D_z & \mathcal{D}_{\bar{z}} & 0 & +i\chi_{\bar{z}\uparrow}^+ \\ D_z\chi_{\bar{z}\uparrow}^- - \frac{1}{2}\chi_{\bar{z}\uparrow}^- D_z & 0 & \mathcal{D}_{\bar{z}} & -i\chi_{\bar{z}\uparrow}^- \\ D_z A_{\bar{z}} & -2iD_z\chi_{\bar{z}\uparrow}^- + 2i\chi_{\bar{z}\uparrow}^- D_z & 2iD_z\chi_{\bar{z}\uparrow}^+ - 2i\chi_{\bar{z}\uparrow}^+ D_z & \partial_{\bar{z}} \end{pmatrix} \quad (3.7)$$

whereas  $\mu_{i\bar{z}}^z, \nu_j^\pm, \omega_{k\bar{z}}$  are tangent vectors of the  $N=2$  moduli space with coordinates  $t^i, \zeta_{\bar{z}\uparrow}^{j\pm}, C^k$ .

The relevant Jacobian is

$$\begin{aligned} \text{sdet}'\mathcal{P} &= \int [dcdb][d\gamma^\pm d\beta^\mp][d\tilde{c}d\tilde{b}] e^{-S_{gh}[c,b,\gamma^\pm,\beta^\mp,\tilde{c},\tilde{b}]} \\ &\times \prod_i \langle \mu_i | b \rangle \prod_k \langle \omega_k | \tilde{b} \rangle \prod_j \delta(\langle \nu_j^+ | \beta^- \rangle) \prod_j \delta(\langle \nu_j^- | \beta^+ \rangle) , \end{aligned} \quad (3.8)$$

where the ghost action

$$S_{gh} = \frac{1}{\pi} \int d^2z \sqrt{g} \mathcal{L}_{gh} \quad (3.9)$$

is obtained by sandwiching  $\mathcal{P}$  between the ghost triple  $(c, \gamma^\pm, \tilde{c})$  and the antighost triple  $(b, \beta^\mp, \tilde{b})$  dual to  $(h_{\bar{z}}^z, \delta\chi_{\bar{z}\uparrow}^\pm, \delta A_{\bar{z}})$ . One finds

$$\begin{aligned} \mathcal{L}_{gh} &= bD_{\bar{z}}c + \beta^- D_{\bar{z}}\gamma^+ + \beta^+ D_{\bar{z}}\gamma^- + \tilde{b}\partial_{\bar{z}}\tilde{c} \\ &+ \chi_{\bar{z}\uparrow}^- [-4b\gamma^+ + D_z(c\beta^+) + \frac{1}{2}\beta^+ D_z c - \beta^+ \tilde{c} - 2D_z\tilde{b}\gamma^+ - 4\tilde{b}D_z\gamma^+] \\ &+ [+4b\gamma^- - D_z(c\beta^-) - \frac{1}{2}\beta^- D_z c - \beta^- \tilde{c} - 2D_z\tilde{b}\gamma^- - 4\tilde{b}D_z\gamma^-] \chi_{\bar{z}\uparrow}^+ \\ &- iA_{\bar{z}}[\beta^- \gamma^+ - \beta^+ \gamma^- - D_z(c\tilde{b})] - (4\tilde{b}\chi_{\bar{z}\uparrow}^- \chi_{\bar{z}\uparrow}^+ D_z \tilde{c}) + \text{tot. der.} \end{aligned} \quad (3.10)$$

The prime on the super determinant in (3.8) indicates that the kernel of the adjoint operator  $\mathcal{P}^\dagger$  is to be projected out, since it consists of the modes outside the image of  $\mathcal{P}$ . The projection is achieved by the antighost insertions in the integrand of (3.8). The kernel of  $\mathcal{P}$  itself spans the isometries of the super-worldsheet and must be compensated for by vertex operator insertions. For a locally flat gauge ( $A = \chi = 0, e = 1$ ) one obtains

$$\mathcal{L}_{gh}^{fix} = b\partial_{\bar{z}}c + \beta^- \partial_{\bar{z}}\gamma^+ + \beta^+ \partial_{\bar{z}}\gamma^- + \tilde{b}\partial_{\bar{z}}\tilde{c}. \quad (3.11)$$

Collecting all results we get the contribution of a fixed topology to the partition function of the  $N=2$  string,

$$\mathcal{Z} = \int \frac{[de_m^a][d\chi_m^\pm][dA_m]}{\text{symmetry volume}} [dZ][d\Psi] e^{-S_m[e,\chi^\pm,A;Z,\Psi]}$$

$$\begin{aligned}
&= \int \left| dt d\zeta^\pm dC \right|^2 \int \left| [dcdb][d\gamma^\pm d\beta^\mp][d\tilde{c}d\tilde{b}] \right|^2 [dZ][d\Psi] e^{-S_m^{fix} - S_{gh}^{fix} - \bar{S}_{gh}^{fix}} \\
&\quad \times \left| \prod_i \langle \mu_i | b \rangle \prod_k \langle \omega_k | \tilde{b} \rangle \prod_j \delta(\langle \nu_j^+ | \beta^- \rangle) \prod_j \delta(\langle \nu_j^- | \beta^+ \rangle) \right|^2.
\end{aligned} \tag{3.12}$$

## 4 Superconformal Algebra, Spectral Flow and BRST

Having derived the complete (matter plus ghost) action away from a locally flat gauge, we are able to calculate the ghost-extended symmetry currents that form the anomaly-free  $N=2$  superconformal constraint algebra,

$$\hat{T}_{ab} = -\frac{1}{\sqrt{g}} \eta_{ac} e_m^c \frac{\delta \sqrt{g} \mathcal{L}}{\delta e_m^b}, \quad \hat{G}_{a\uparrow\downarrow}^\pm = -\frac{1}{4} \frac{\delta \mathcal{L}}{\delta \chi_{a\uparrow\downarrow}^\mp}, \quad \hat{J}_a = -\frac{i}{4} \frac{\delta \mathcal{L}}{\delta A_a}. \tag{4.1}$$

In a locally flat gauge these currents factorize holomorphically, with

$$\begin{aligned}
\hat{T}_{zz} &= -\frac{1}{2} \partial_z Z^{-\mu} \partial_z Z^+_{\mu} - \frac{1}{4} \partial_z \Psi_{\downarrow}^{-\mu} \Psi_{\downarrow}^+_{\mu} - \frac{1}{4} \partial_z \Psi_{\downarrow}^{+\mu} \Psi_{\downarrow}^{-\mu} \\
&\quad + 2\partial cb + c\partial b + \partial \tilde{c} \tilde{b} - \frac{3}{2} (\partial \gamma^- \beta^+ + \partial \gamma^+ \beta^-) - \frac{1}{2} (\gamma^- \partial \beta^+ + \gamma^+ \partial \beta^-), \\
\hat{G}_{z\downarrow}^\pm &= \partial_z Z^{\mp\mu} \Psi_{\downarrow}^\pm_{\mu} \\
&\quad - 4\gamma^\pm b \mp 4\partial \gamma^\pm \tilde{b} \mp 2\gamma^\pm \partial \tilde{b} + \frac{3}{2} \partial c \beta^\pm + c \partial \beta^\pm \mp \tilde{c} \beta^\pm, \\
\hat{J}_z &= -\frac{1}{2} \Psi_{\downarrow}^{-\mu} \Psi_{\downarrow}^+_{\mu} \\
&\quad + \partial(c\tilde{b}) + (\gamma^+ \beta^- - \gamma^- \beta^+),
\end{aligned} \tag{4.2}$$

decomposing into matter and ghost parts, e.g.  $\hat{T} = T_m + T_{gh}$ . Canonical quantization of the  $N=2$  string is achieved by imposing the usual operator product expansions for the matter fields,

$$Z^{\pm\mu}(z) Z^{\mp\nu}(w) \sim -2\eta^{\mu\nu} \log(z-w), \quad \Psi_{\downarrow}^{\pm\mu}(z) \Psi_{\downarrow}^{\mp\nu}(w) \sim -2\eta^{\mu\nu} (z-w)^{-1}, \tag{4.3}$$

and for the ghost fields,

$$\beta^\pm(z) \gamma^\mp(w) \sim \frac{-1}{z-w}, \quad b(z) c(w) \sim \frac{1}{z-w}, \quad \tilde{b}(z) \tilde{c}(w) \sim \frac{1}{z-w}. \tag{4.4}$$

The operator product algebra is then given by

$$\begin{aligned}
\hat{T}(z) \hat{T}(w) &\sim \frac{c}{2} (z-w)^{-4} + 2(z-w)^{-2} \hat{T}(w) + (z-w)^{-1} \partial \hat{T}(w), \\
\hat{T}(z) \hat{G}^\pm(w) &\sim \frac{3}{2} (z-w)^{-2} \hat{G}^\pm(w) + (z-w)^{-1} \partial \hat{G}^\pm(w), \\
\hat{T}(z) \hat{J}(w) &\sim (z-w)^{-2} \hat{J}(w) + (z-w)^{-1} \partial \hat{J}(w), \\
\hat{G}^+(z) \hat{G}^-(w) &\sim \frac{4c}{3} (z-w)^{-3} - 4(z-w)^{-2} \hat{J}(w) + 4(z-w)^{-1} (\hat{T}(w) - \frac{1}{2} \partial \hat{J}(w)),
\end{aligned}$$



$$\begin{aligned}
\hat{G}^\pm(z) \hat{G}^\pm(w) &\sim 0, \\
\hat{J}(z) \hat{G}^\pm(w) &\sim \mp(z-w)^{-1} \hat{G}^\pm(w), \\
\hat{J}(z) \hat{J}(w) &\sim \frac{c}{3}(z-w)^{-2}.
\end{aligned} \tag{4.5}$$

with central charge  $c = c_m + c_{gh} = 6 - 6 = 0$ .

The superconformal algebra (4.5) possesses a continuous automorphism termed spectral flow [23]. It can be generated by an operator which is constructed out of the  $U(1)$  gauge current  $J$  as <sup>4</sup>

$$SFO(\Theta) = \exp\left[\Theta \oint \frac{dz}{2\pi i} \log(z) J(z)\right], \quad SFO(\Theta)^{-1} = SFO(-\Theta). \tag{4.6}$$

and acts on the currents as follows:

$$\begin{aligned}
T(z) &\rightarrow SFO(\Theta)T(z)SFO(\Theta)^{-1} = T(z) + \frac{\Theta}{z}J(z) + \frac{c}{6}\frac{\Theta^2}{z^2}, \\
G^\pm(z) &\rightarrow SFO(\Theta)G^\pm(z)SFO(\Theta)^{-1} = z^{\mp\Theta}G^\pm(z), \\
J(z) &\rightarrow SFO(\Theta)J(z)SFO(\Theta)^{-1} = J(z) + \frac{c}{3}\frac{\Theta}{z}.
\end{aligned} \tag{4.7}$$

The definition of  $SFO$  should be understood as a successive application of operator products, with the integrals of  $J$  being rather formal. For example,

$$\begin{aligned}
T(w) &\rightarrow T(w) + \Theta \oint \frac{dz}{2\pi i} \log(z) J(z) T(w) + \frac{\Theta^2}{2} \oint \frac{dz'}{2\pi i} \log(z') J(z') \oint \frac{dz}{2\pi i} \log(z) J(z) T(w) \\
&= T(w) + \Theta \oint \frac{dz}{2\pi i} \log(z) \frac{J(w)}{(z-w)^2} + \frac{\Theta^2}{2} \oint \frac{dz'}{2\pi i} \log(z') J(z') \oint \frac{dz}{2\pi i} \log(z) \frac{J(w)}{(z-w)^2} \\
&= T(w) + \frac{\Theta}{w} J(w) + \frac{c\Theta^2}{6} \oint \frac{dz'}{2\pi i} \log(z') J(z') \frac{J(w)}{w} \\
&= T(w) + \frac{\Theta}{w} J(w) + \frac{c}{6} \frac{\Theta^2}{w^2}.
\end{aligned} \tag{4.8}$$

For the hatted currents one finds of course  $c = 0$ . A variant of this  $SFO$  will arise automatically in the evaluation of the path integral for the  $N=2$  string. Similar considerations apply independently for the anti-holomorphic copy of the superconformal algebra.

The same ghost-extended symmetry currents (4.2) may be calculated by using the canonical quantization procedure and BRST methods. The resulting (holomorphic part of the) BRST current is given by

$$\begin{aligned}
J_{BRST} &= cT + \gamma^+ G^- + \gamma^- G^+ + \tilde{c}J + c\partial cb + c\partial\tilde{c}\tilde{b} \\
&\quad - 4\gamma^+ \gamma^- b + 2\partial\gamma^- \gamma^+ \tilde{b} - 2\partial\gamma^+ \gamma^- \tilde{b} \\
&\quad + \frac{3}{4}\partial c(\gamma^+ \beta^- + \gamma^- \beta^+) - \frac{3}{4}c(\partial\gamma^+ \beta^- + \partial\gamma^- \beta^+) \\
&\quad + \frac{1}{4}c(\gamma^+ \partial\beta^- + \gamma^- \partial\beta^+) + \tilde{c}(\gamma^+ \beta^- - \partial\gamma^- \beta^+),
\end{aligned} \tag{4.9}$$

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<sup>4</sup> For a derivation see Appendix B.

and the BRST charge is defined by

$$Q = \oint \frac{dz}{2\pi i} J_{BRST}. \quad (4.10)$$

The ghost-extended currents then obey

$$\hat{T}(w) = \{Q, b(w)\}, \quad \hat{G}^\pm(w) = [Q, \beta^\pm(w)], \quad \hat{J}(w) = \{Q, \tilde{b}(w)\}. \quad (4.11)$$

For completeness we list the BRST transformations of all matter and ghost fields in the superconformal gauge,

$$\begin{aligned} [Q, Z^{\pm\mu}] &= c\partial Z^{\pm\mu} - 2\gamma^\mp \Psi^{\pm\mu}, \\ \{Q, \Psi^{\pm\mu}\} &= c\partial \Psi^{\pm\mu} + \frac{1}{2}\partial c \Psi^{\pm\mu} - 2\gamma^\pm \partial Z^{\pm\mu} \mp \tilde{c} \Psi^{\pm\mu}, \\ \{Q, c\} &= c\partial c - 4\gamma^- \gamma^+, \\ [Q, \gamma^\pm] &= c\partial \gamma^\pm - \frac{1}{2}\partial c \gamma^\pm \mp \tilde{c} \gamma^\pm, \\ \{Q, \tilde{c}\} &= c\partial \tilde{c} - 2\gamma^- \partial \gamma^+ + 2\gamma^+ \partial \gamma^-. \end{aligned} \quad (4.12)$$

The invariant (Neveu-Schwarz) vacuum is defined by the vanishing of all  $N=2$  currents at the origin  $z = 0$ , which translates to

$$\hat{L}_{n \geq -1} |0\rangle = 0, \quad \hat{G}_{r \geq -1/2} |0\rangle = 0, \quad \hat{J}_{n \geq 0} |0\rangle = 0. \quad (4.13)$$

The subalgebra of (4.5) which leaves the vacuum invariant is the  $SL(2, \mathbb{R}) \times U(1)$  generated by  $\hat{L}_{\pm 1}$ ,  $\hat{L}_0$  and  $\hat{J}_0$ . Again, one ultimately has to adjoin the antiholomorphic sector as well.

## 5 The (2,2) Moduli Space and Instanton Background

The space of moduli is defined as the complement to the range of gauge transformations. Since a gauge transformation is usually mediated by a differential operator  $D$  and a transformation parameter  $\xi$ , one arrives at

$$m \in \{\text{moduli}\} \quad \Leftrightarrow \quad 0 = \langle m | D\xi \rangle = \langle D^\dagger m | \xi \rangle \quad \forall \xi \quad \Leftrightarrow \quad m \in \ker(D^\dagger). \quad (5.1)$$

The most powerful tool for the investigation of the  $N=2$  moduli space is the Atiyah-Singer index theorem. With this theorem it is possible to determine the dimensions of the kernels of differential operators like in (5.1). For a Lorentz and gauge covariant derivative  $\mathcal{D}_{\bar{z}}^{n,q} = \partial_{\bar{z}} - n\omega_{\bar{z}} + 2iqA_{\bar{z}}$  on the space of  $(-n)$ -forms over a compact Riemann surface with  $p$  punctures one obtains the relation ( $n \in \frac{1}{2}\mathbb{Z}$ ,  $q \in \mathbb{Z}$ )

$$\dim \ker(\mathcal{D}_{\bar{z}}^{n,q\dagger}) - \dim \ker(\mathcal{D}_{\bar{z}}^{n,q}) = (2n+1)(g-1) + 2qc + p, \quad (5.2)$$

where  $g \in \mathbb{Z}_+$  is the genus of the Riemann surface and  $c \in \mathbb{Z}$  is the first Chern number (instanton number) of the  $U(1)$ -bundle (not to be confused with the diffeomorphism ghost

or the central charge). Generically, one of the two kernels is empty, depending on the sign of the r.h.s. of (5.2). In case  $p = 0$  and  $q = 0$ , an additional zero mode will occur simultaneously in both kernels for  $(n = 0, g \geq 1)$  or for (any  $n, g = 1$ , odd spin structure).

For the  $N=2$  string the appropriate weights  $n$  and charges  $q$  should be taken from the diffeomorphisms  $\xi^z$ , the susy transformations  $\epsilon^\pm$  and the gauge transformation  $\alpha$ . For a genus  $g > 1$  and  $p$  punctures one gets

$n = 1, q = 0:$	$3g - 3 + p$	metric moduli $\mu$
$n = \frac{1}{2}, q = \pm \frac{1}{2}:$	$2g - 2 \pm c + p$	fermionic moduli $\nu^\pm$
$n = 0, q = 0:$	$g - 1 + p + \delta_{p,0}$	gauge moduli $\omega$

For a Chern number  $|c| > 2(g-1) + p$ , one index of (5.2) turns negative, due to additional zero modes of  $\gamma^+$  or  $\gamma^-$ . In the path integral, these additional zero modes cannot be compensated for and let the path integral vanish [7] outside the range of

$$|c| \leq 2(g-1) + p. \quad (5.3)$$

The dimensionalities given above are complex, but moduli space has a complex structure, allowing one to consider left- and right-moving moduli independently.

Let us investigate the properties of the fermionic moduli in more detail. As was shown in Ref.[22] it is possible to choose a special basis in fermionic moduli space which has a pointlike support. In that way we are able to parametrize the (variation of the) gauge-fixed gravitini as

$$\delta\chi_{\bar{z}\uparrow}^\pm = \chi_{\bar{z}\uparrow}^\pm = \sum_{j=1}^{2g-2\pm c+p} \zeta_{\bar{z}\uparrow}^{\pm j} \delta^2(z-z_j^\pm), \quad \text{i.e. } \nu^{\pm j} = \delta^2(z-z_j^\pm). \quad (5.4)$$

With different points of support for  $\chi_{\bar{z}\uparrow}^+$  and  $\chi_{\bar{z}\uparrow}^-$ , the annoying term quadratic in  $\chi_{\bar{z}\uparrow}^\pm$  vanishes from the ghost action. Furthermore, the expression (5.4) can be inserted into the path integral (3.12). The finite-dimensional integral over the fermionic coordinates  $\zeta^\pm$  is carried out rather easily [24] and yields ( $p = 0$ )

$$\begin{aligned} \mathcal{Z}(g, c) = & \int |dt dC|^2 \int |dc db| [d\gamma^\pm d\beta^\mp] [d\tilde{c} d\tilde{b}]^2 [dZ] [d\Psi] e^{-S^{fix}[t, C; Z, \Psi, c, b, \gamma^\pm, \beta^\mp, \tilde{c}, \tilde{b}]} \\ & \times \left| \prod_{i=1}^{3g-3} \langle \mu_i | b \rangle \prod_{k=1}^g \langle \omega_k | \tilde{b} \rangle \prod_{j=1}^{2g-2+c} \delta(\beta^-(z_j^+)) \hat{G}^-(z_j^+) \prod_{j=1}^{2g-2-c} \delta(\beta^+(z_j^-)) \hat{G}^+(z_j^-) \right|^2 \end{aligned} \quad (5.5)$$

The combination  $\delta(\beta)\hat{G}$  can be identified with the picture-changing operators [25, 11]

$$PCO^\pm(z) = \{Q, H(\beta^\pm(z))\} = \delta(\beta^\pm(z))\{Q, \beta^\pm(z)\} = \delta(\beta^\pm(z))\hat{G}^\pm(z) \quad , \quad (5.6)$$

where  $H$  denotes the Heaviside step function. String scattering amplitudes ( $p > 0$ ) are obtained from (5.5) by shifting the number of moduli according to the table above and by inserting a string of  $p$  vertex operators in their canonical representation (discussed in Section 7). In any case, the fermionic moduli are responsible for the correct number

of picture-changing operator insertions needed to balance the zero modes of the  $U(1)$  charged fields.

A new ingredient compared to the  $N=1$  string is the gauged  $U(1)$  symmetry and its moduli space which should be treated with care. First of all,  $U(1)$  bundles are topologically classified by the integral Chern number

$$c = \frac{1}{2\pi} \int F, \quad F = dA \quad \text{locally} \quad (5.7)$$

which we have already encountered in the index theorem. Since the space of connections is an affine space, it is possible to split the  $U(1)$  connection  $A$  into a topologically trivial one,  $A_0$  which is integrated over in the path integral, and a fixed background connection,  $A_c$  which yields the Chern number,

$$A = A_c + A_0, \quad \text{with} \quad \int F_c = 2\pi c \quad \text{and} \quad \int F_0 = 0. \quad (5.8)$$

All continuous degrees of freedom are left in  $A_0$  which, as a globally defined one-form, may be Hodge-decomposed into exact, co-exact and harmonic parts,

$$A_0 = d\lambda + i * d\mu + h. \quad (5.9)$$

A peculiarity of  $N=2$  string theory is the fact that the connection  $A$  transforms under a  $U_V(1)$  with parameter  $\alpha$  as well as under a  $U_A(1)$  with parameter  $\hat{\alpha}$  (see (2.8)),

$$\begin{aligned} A^\alpha &= e^{-i\alpha}(A + id)e^{i\alpha} = A - d\alpha, & F^\alpha &= F, \\ A^{\hat{\alpha}} &= e^{-i\hat{\alpha}}(A - *d)e^{i\hat{\alpha}} = A - i * d\hat{\alpha} & F^{\hat{\alpha}} &= F - *\Delta\hat{\alpha}. \end{aligned} \quad (5.10)$$

With these two transformations the exact and the co-exact parts in (5.9) can be completely eliminated so that only the harmonic contribution remains. In other words, one may choose a gauge ( $\lambda = \phi = 0$ ) where

$$A_0 = h \quad \implies \quad F_0 = 0 = d * A_0, \quad (5.11)$$

a flat connection in the Lorentz gauge.

On a compact punctured Riemann surface there exists a  $2(g+p-1)$  dimensional basis of real harmonic one-forms  $\alpha_i, \beta_i$ ,  $i = 1, \dots, g$ , and  $\gamma_\ell, \delta_\ell$ ,  $\ell = 1, \dots, p-1$ , dual to a homology basis of cycles  $a_i, b_i$  and  $c_\ell, d_\ell$ . Hence, an element of the  $U(1)$  Teichmüller space may be written as

$$h = 2\pi(A^i \alpha_i + B^i \beta_i + G^\ell \gamma_\ell + D^\ell \delta_\ell) \quad \text{with} \quad \vec{A}, \vec{B} \in \mathbb{R}^g \quad \text{and} \quad \vec{G}, \vec{D} \in \mathbb{R}^{p-1}. \quad (5.12)$$

Note that the  $\ell$  sum runs only to  $p-1$ , because the sum of all puncture holonomies must vanish. The coefficient vectors  $\vec{A}, \vec{B}, \vec{G}, \vec{D}$  can take any real values. When all are integral, however,  $g(z) = \exp\{i \int_p^z h\}$  is single-valued and generates a “large” gauge transformation not connected to the identity. Dividing out these “modular” gauge transformations compactifies the  $U(1)$  Teichmüller space  $\mathbb{R}^{2(g+p-1)}$  to the moduli space  $(\mathbb{R}/\mathbb{Z})^{2(g+p-1)}$ , since all

coefficients become periodic on  $[0, 1)$ . It is convenient to combine the twists to  $(g+p-1)$ -dimensional vectors  $\tilde{A} = (\vec{A}, \vec{G})$  and  $\tilde{B} = (\vec{B}, \vec{D})$  and rewrite the flat connection in terms of a vector  $\Omega$  of meromorphic one-forms  $\omega_i$  and  $\omega_{P_i P_{i+1}}$ ,

$$\begin{aligned} h &= 2\pi(\tilde{B} - \bar{T}\tilde{A})(T - \bar{T})^{-1}\Omega - 2\pi(\tilde{B} - T\tilde{A})(T - \bar{T})^{-1}\bar{\Omega} \\ &= -i\pi\bar{C}(\text{Im}T)^{-1}\Omega + i\pi C(\text{Im}T^{-1})\bar{\Omega} \quad \text{with} \quad C = \tilde{B} - T\tilde{A} \in \mathbb{C}^{g+p-1}/\Lambda, \end{aligned} \quad (5.13)$$

using the generalized period matrix  $T$  and Jacobian lattice  $\Lambda$ . As a sum of a meromorphic and an anti-meromorphic piece,  $h$  has simple poles at the punctures, with residues given by  $(G^\ell - G^{\ell-1}, D^\ell - D^{\ell-1})$  for  $\ell = 1, \dots, p-1$  and  $(G^p, D^p) = (-G^{p-1}, -D^{p-1})$ , making their sum vanish. Appendix C contains further details. Some care is required with the puncture (co)homologies. The punctured surface may be regarded as a degeneration limit of an unpunctured surface of genus  $g+p-1$ . In this limit,  $p-1$  harmonic one-forms  $\delta_\ell$  formally disappear since the  $d_\ell$  cycles blow up. The meromorphic one-forms built with them, however, remain finite. Although their contributions to  $h$  in (5.13) vanish, as diagonal elements of  $\text{Im}T$  diverge, their singular integrals along  $d_\ell$  produce finite twists  $D^\ell$ . Note that in the holomorphic basis the spectral flow acts on the gauge puncture moduli by simple shift, and holomorphic factorization is manifest.

Let us consider some possible background configurations  $A_c$  explicitly. For this purpose it will be sufficient to consider a compact Riemann surface of genus  $g$  without any punctures ( $p = 0$ ). A non-vanishing Chern number  $c$  means that we have to introduce more than one coordinate patch on the Riemann surface, with non-trivial transition functions in their overlap. To be more precise, only the  $U_V(1)$ -bundle may be non-trivial. The  $U_A(1)$  transition functions must stay trivial in order to have a well-defined Chern number. This is markedly different from [8, 9, 14] where formally *two* Chern numbers were introduced, one for the holomorphic and one for the anti-holomorphic half of the theory. Such a prescription would in fact redefine the  $(2, 2)$  string, in analogy to the GSO projection of the  $(1, 1)$  string, where in the sum over spin structures one replaces the *common* holonomies for left- and right-movers by *independent* ones. Since spectral flow can be generalized to change the Chern number (see below), it is natural to introduce left- and right-moving Chern numbers, at the expense of giving up the local worldsheet description. We shall see, however, that there is no need for a GSO projection in the  $N=2$  string, and one therefore has a choice here. In the present work we will not follow [8, 9, 14], but keep the geometrical path-integral definition.

Our requirement for a trivial  $U_A(1)$  bundle may be compared with the nature of diffeomorphisms and Weyl transformations. In order to have a well-defined genus, one has to require a globally defined Weyl parameter  $\sigma$  so its contribution to the curvature scalar  $\mathcal{R}$  is a total derivative. The same applies for  $c$  and  $U_A(1)$ ,

$$-8\pi(g-1) = \int \sqrt{g}\mathcal{R}^\sigma = \int \sqrt{g}(\mathcal{R} + \Delta\sigma), \quad 2\pi c = \int F^{\hat{\alpha}} = \int (F - *\Delta\hat{\alpha}). \quad (5.14)$$

For that reason we only have to consider non-trivial  $U_V(1)$  transition functions for the construction of some  $A_c$ .

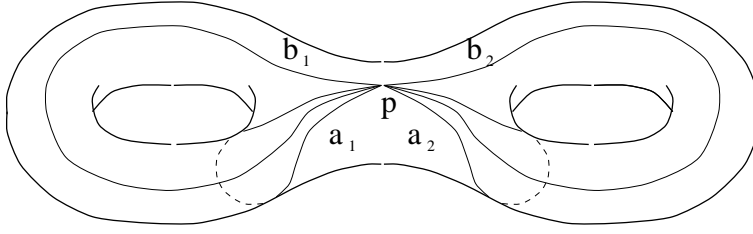


Figure 1: Genus 2 Riemann surface.

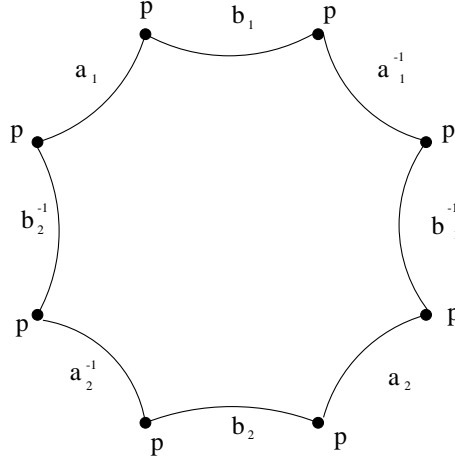


Figure 2: Cut genus 2 Riemann surface.

One way to cover a compact Riemann surface with coordinate patches is to cut it along a homology basis  $a_i, b_i$ , demonstrated in Figure 1. By this procedure one obtains a simply-connected region with a boundary as given in Figure 2. The edges of this region,  $a_i, a_i^{-1}$  respectively  $b_i, b_i^{-1}$ , have to be identified in order to reconstruct the original surface. On these edges one may install the non-trivial transition functions of the  $U_V(1)$  bundle.

For practical reasons we choose the Lorentz gauge for the connection,

$$d * A = \partial_z A_{\bar{z}} + \partial_{\bar{z}} A_z = 0. \quad (5.15)$$

The possible transition functions  $g = e^{i\alpha}$  are then restricted to have a harmonic exponent,

$$\partial_z \partial_{\bar{z}} \alpha = 0 \quad \Longleftrightarrow \quad \alpha(z, \bar{z}) = \alpha(z) + \bar{\alpha}(\bar{z}). \quad (5.16)$$

The form of a finite  $U_V(1)$  transition function  $g$  on the homology cycles  $a_i, b_i$  can be built from the harmonic one-forms  $\alpha_i, \beta_i$ ,

$$\begin{aligned} g_{a_i} &= \exp\left\{-2\pi i \int_p^z \sum_{j=1}^g (\alpha_j u_{ji}^a + \beta_j v_{ji}^a)\right\}, \\ g_{b_i} &= \exp\left\{-2\pi i \int_p^z \sum_{j=1}^g (\alpha_j u_{ij}^b + \beta_j v_{ij}^b)\right\}, \end{aligned} \quad (5.17)$$

where  $g_{a_i}, g_{b_i}$  live on the cycles  $a_i, b_i$  which have to be crossed when walking around a cycle  $b_i, a_i$ . If one requires a transition function of this type to be single-valued on the *whole* Riemann surface, one has to identify all coefficients  $u, v$  on different cycles and take them to be integral, which yields just a “large” gauge transformation.

A simple choice of a connection and field strength with Chern number  $c$  employs only the harmonic forms  $\alpha_i, \beta_i$ , and is given by

$$A_c = \frac{\pi ic}{2g} [\omega (\text{Im}\tau)^{-1} \int_p^z \bar{\omega} - \bar{\omega} (\text{Im}\tau)^{-1} \int_p^z \omega] \quad \Longrightarrow \quad F_c = -\frac{\pi ic}{g} \omega \wedge (\text{Im}\tau)^{-1} \bar{\omega}. \quad (5.18)$$

For the holonomies of  $A_c$  we get

$$\begin{aligned} A_c(z + a_k) - A_c(z) &= \frac{\pi ic}{2g} (\omega_i + \bar{\omega}_i) (\text{Im}\tau)_{ik}^{-1} = ig_{b_k}^{-1} dg_{b_k}, \\ A_c(z + b_k) - A_c(z) &= \frac{\pi ic}{2g} (\omega_i (\text{Im}\tau)_{ij}^{-1} \bar{\tau}_{jk} - \bar{\omega}_i (\text{Im}\tau)_{ij}^{-1} \tau_{jk}) = ig_{a_k}^{-1} dg_{a_k} \end{aligned} \quad (5.19)$$

where  $z + a_k, b_k$  denotes a shift of  $z$  along that homology cycle. The parameters  $u, v$  in (5.17) are then specified via

$$\begin{aligned} g_{b_k} &= \exp \left[ \frac{\pi c_1}{2g} e_k (\text{Im}\tau)^{-1} \left( \int_p^z \omega - \int_p^z \bar{\omega} \right) \right], \\ g_{a_k} &= \exp \left[ \frac{\pi c_1}{2g} e_k (\bar{\tau} (\text{Im}\tau)^{-1} \int_p^z \omega - \tau (\text{Im}\tau)^{-1} \int_p^z \bar{\omega}) \right], \\ e_k &= (0, \dots, 0, 1, 0, \dots, 0) \quad k\text{th unit vector of } \mathbb{R}^g. \end{aligned} \quad (5.20)$$

With these fixed parameters, the transition functions compensate the holonomies (5.19), and they are single-valued on the *boundary* of the cut Riemann surface when  $c \in \mathbb{Z}$ .

Another convenient choice for the gauge background field strength is a  $\delta$ -distributed two-form which has some calculational advantages but whose connection cannot be given explicitly. To construct such a connection one has to use meromorphic one-forms  $m$  with simple poles at the punctures [26] (see Appendix C). To obtain a  $\delta$ -distributed  $U(1)$  field strength one makes use of

$$\partial_{\bar{z}} \frac{1}{z} = \partial_z \frac{1}{\bar{z}} = 2\pi \delta^2(z). \quad (5.21)$$

However, since the residues of a globally defined meromorphic one-form always sum to zero,

$$\sum_P \text{res}_P(m) = 0, \quad (5.22)$$

one has to glue together at least two different one-forms by a  $U(1)$ -transformation. Let us take a meromorphic one-form  $m = \omega_{PQ}$  with properties

$$\text{res}_P(\omega_{PQ}) = -\text{res}_Q(\omega_{PQ}) = 1 \quad \text{and} \quad \oint_{a_k} \omega_{PQ} = 0 \quad k = 1, \dots, g \quad (5.23)$$

and a holomorphic one-form  $\omega$ . Let us further choose a disc  $\mathcal{D}$  on the Riemann surface  $\mathcal{M}$  such that

$$P \in \mathcal{D}, \quad Q \in \mathcal{M} \setminus \mathcal{D}.$$

Now we define two connection one-forms restricted on the two sets  $\mathcal{D}$  and  $\mathcal{M}\setminus\mathcal{D}$  by

$$\begin{aligned} A_c &= \frac{c}{2i} (\omega_{PQ} - \bar{\omega}_{PQ}) \quad \text{on } \mathcal{D}, \\ A'_c &= (\omega + \bar{\omega}) \quad \text{on } \mathcal{M}\setminus\mathcal{D}. \end{aligned} \quad (5.24)$$

These two connections can be patched together by a  $U(1)$ -transformation  $d\Lambda = ig^{-1}dg$  which has to satisfy the cocycle condition

$$\oint_P d\Lambda = \oint_P \left( \frac{c}{2i} \omega_{PQ} - \omega + c.c. \right) = 2\pi c \in 2\pi\mathbb{Z}.$$

The corresponding field strength then reads

$$F_c = 2\pi ic \delta^2(z - z_P) dz \wedge d\bar{z}, \quad (5.25)$$

and the quantization of the Chern number follows from the cocycle condition.

## 6 Bosonization in a Gauge Background

In order to understand the consequences of the gauge moduli in the  $N=2$  string path integral, bosonization of the charged fields  $\Psi^{\pm\mu}, \beta^{\pm}, \gamma^{\mp}$  proves to be very useful. The bosonized formulation facilitates most explicit calculations. Let us investigate the changes in the bosonization method due to the additional  $U(1)$  gauge coupling in the  $N=2$  string action. We keep the superconformal gauge but make explicit the dependence on the gauge field moduli and topology.

It is convenient to first consider only the matter fields  $\Psi_{\downarrow}^{\pm}, \Psi_{\uparrow}^{\pm}$ . Since  $n = -\frac{1}{2}$ , the index theorem (5.2) tells us that a background charge can only come from a non-trivial gauge bundle. The index  $\mu$  will be suppressed temporarily. The part of the action involving a single complex Majorana fermion  $\Psi$  reads

$$\begin{aligned} S_{\Psi} = -\frac{1}{8\pi} \int d^2z \Big[ & \Psi_{\downarrow}^{-} (\partial_{\bar{z}} + iA_{\bar{z}}) \Psi_{\downarrow}^{+} + \Psi_{\downarrow}^{+} (\partial_{\bar{z}} - iA_{\bar{z}}) \Psi_{\downarrow}^{-} \\ & + \Psi_{\uparrow}^{-} (\partial_z + iA_z) \Psi_{\uparrow}^{+} + \Psi_{\uparrow}^{+} (\partial_z - iA_z) \Psi_{\uparrow}^{-} \Big], \end{aligned} \quad (6.1)$$

and we go to the flat gauge,

$$A = A_z dz + A_{\bar{z}} d\bar{z} = A_c + h = A_c(z, \bar{z}) + h_z(z) dz + h_{\bar{z}}(\bar{z}) d\bar{z}, \quad (6.2)$$

with  $A_c$  and  $h$  given by (5.24) and (5.13), respectively. Of course, we could have gauged  $A = 0$  completely in any simply-connected region, and so the (local) bosonization relations should not feel the presence of the moduli. Hence, just as for the metric moduli, we shall proceed by first ignoring  $A$  and later accounting for its non-trivial holonomy and topology.

Because  $J_V = *J_A$ , we cannot simultaneously bosonize both  $U(1)$  gauge currents, as  $d\Phi_V = *d\Phi_A$  implies that the two bosons would not be mutually local. We therefore have



to make a choice and pick the  $U_V(1)$ , which rotates left- and right-movers in the same way. With  $\Psi_{\downarrow}$  locally holomorphic and  $\Psi_{\uparrow}$  locally anti-holomorphic on-shell, we write

$$\Psi_{\downarrow}^{\pm}(z) = i\sqrt{2}e^{\pm\phi}(z) \quad \text{and} \quad \Psi_{\uparrow}^{\pm}(\bar{z}) = -i\sqrt{2}e^{\pm\bar{\phi}}(\bar{z}), \quad (6.3)$$

with chiral bosons  $\phi(z)$  and  $\bar{\phi}(\bar{z})$  obeying

$$\phi(z) \phi(w) \sim \log(z-w) \quad \text{and} \quad \bar{\phi}(\bar{z}) \bar{\phi}(\bar{w}) \sim \log(\bar{z}-\bar{w}). \quad (6.4)$$

The  $U(1)$  current  $J = J_z dz + J_{\bar{z}} d\bar{z}$  becomes <sup>5</sup>

$$\begin{aligned} J_z &= \frac{1}{2} \Psi_{\downarrow}^+ \Psi_{\downarrow}^- = -\partial_z \phi = -\partial_z \Phi, \\ J_{\bar{z}} &= \frac{1}{2} \Psi_{\uparrow}^+ \Psi_{\uparrow}^- = +\partial_{\bar{z}} \bar{\phi} = +\partial_{\bar{z}} \Phi, \end{aligned} \quad (6.5)$$

in accordance with  $*J = id\Phi$ , where the full boson

$$\Phi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z}) \quad (6.6)$$

satisfies  $\partial_{\bar{z}} \partial_z \Phi = 0$  on-shell in regions for vanishing  $F$ . It is instructive to check out reality properties. Under complex conjugation, with the rule  $(XY)^* = Y^* X^*$ , the action (6.1) just changes sign <sup>6</sup> if  $(\Psi_{\downarrow}^{\pm})^* = \Psi_{\uparrow}^{\mp}$ , which implies  $J^* = J$  and  $\Phi^* = -\Phi$  by virtue of  $\phi^* = -\bar{\phi}$ . We choose all nonchiral bosons to be imaginary. One has of course also the trivial symmetry of flipping the  $U(1)$  charges and the signs of all bosons (including  $A$ ).

In complete analogy with the standard case, the bosonized action is

$$S_{\Phi} = -\frac{1}{4\pi} \int d^2 z \left[ \partial_{\bar{z}} \Phi \partial_z \Phi - 2i F_{z\bar{z}} \Phi \right] = -\frac{1}{4\pi} \int \left[ \frac{1}{2} d\Phi \wedge *d\Phi + 2F\Phi \right]. \quad (6.7)$$

The term linear in  $\Phi$  is required in order to agree with the index theorem,

$$4\pi [\sharp(\Psi^+) - \sharp(\Psi^-)] = i \int dJ = \int d*d\Phi = 2i \int d^2 z \partial_{\bar{z}} \partial_z \Phi = -2i \int d^2 z F_{z\bar{z}} = 2 \int F = 4\pi c \quad (6.8)$$

where  $\sharp(\cdot)$  counts the number of zero modes of the field in question, and we used the Minkowskian equations of motion ( $F \rightarrow iF$ ). Since the gauge freedom allows us to concentrate the field strength in an arbitrarily small region, we may choose

$$-\frac{1}{2\pi} \int F\Phi \quad \longrightarrow \quad -c \Phi(P) \quad (6.9)$$

for some point  $P$ . Apparently, the Chern number couples only the zero mode of  $\Phi$ . The latter also controls the conservation of the charge associated to the global subgroup of  $U_V(1)$  in (2.10),

$$\Psi_{\downarrow\uparrow}^{\pm} \rightarrow e^{\pm 2\pi i \rho} \Psi_{\downarrow\uparrow}^{\pm}, \quad \phi \rightarrow \phi + 2\pi i \rho, \quad \bar{\phi} \rightarrow \bar{\phi} + 2\pi i \rho. \quad (6.10)$$

<sup>5</sup> In anti-holomorphic OPEs, anti-radial ordering must be taken for consistency with conjugation.

<sup>6</sup> This is because we work with Euclideanized fermions; in Minkowski space it is invariant.

We take into account the global properties of  $\Phi$ . It is well known [27, 28] that the bosonization procedure employs compact bosons, which carry integral or half-integral winding numbers  $m_k, n_k, p_\ell, q_\ell \in \frac{1}{2}\mathbb{Z}$  around the homology cycles  $a_k, b_k, c_\ell, d_\ell$ , respectively,

$$\begin{aligned}\Phi(z + a_k) - \Phi(z) &= 2\pi i m_k, & \Phi(z + b_k) - \Phi(z) &= 2\pi i n_k, \\ \Phi(z + c_\ell) - \Phi(z) &= 2\pi i p_\ell, & \Phi(z + d_\ell) - \Phi(z) &= 2\pi i q_\ell.\end{aligned}\quad (6.11)$$

Since non-vanishing winding numbers around a puncture are due to terms  $\Phi \sim \log(z\bar{z})$  and  $\Phi \sim \log(z/\bar{z})$  for a coordinate patch centered at the puncture, the bosonic action (6.7) seems to give a divergent contribution and thus to suppress  $\Phi$  configurations with non-trivial puncture windings. One should, however, regulate these contributions by considering a genus  $g+p-1$  unpunctured surface close to degeneration, and take the limit only at the end of the computation. We separate the multi-valued (solitonic) part of the bosons by writing

$$\Phi = \Phi_s + \Phi_q = (\phi_s + \bar{\phi}_s) + (\phi_q + \bar{\phi}_q). \quad (6.12)$$

The split is made unique by demanding  $\Phi_s$  to be harmonic. Again grouping  $\tilde{m} = (\vec{m}, \vec{p})$  and  $\tilde{n} = (\vec{n}, \vec{q})$ , the solitonic function is explicitly given by [27, 28]

$$\begin{aligned}\Phi_s(P) &= 2\pi i(\tilde{n} - \bar{T}\tilde{m})(T - \bar{T})^{-1} \int_{P_0}^P \Omega - 2\pi i(\tilde{n} - T\tilde{m})(T - \bar{T})^{-1} \int_{P_0}^P \bar{\Omega} \\ &= \pi \bar{K}(\text{Im}T)^{-1} \int_{P_0}^P \Omega - \pi K(\text{Im}T)^{-1} \int_{P_0}^P \bar{\Omega} \quad \text{with} \quad K = \tilde{n} - T\tilde{m} \in \frac{1}{2}\Lambda,\end{aligned}\quad (6.13)$$

on which the action (6.7) evaluates as

$$S_\Phi(K) = \frac{\pi}{2} \bar{K}(\text{Im}T)^{-1} K - \frac{1}{4\pi} \int d^2z \partial_{\bar{z}} \Phi_q \partial_z \Phi_q - c \Phi(P). \quad (6.14)$$

One should note that the differential  $d\Phi_s$  is a linear combination of holomorphic and antiholomorphic one-forms and therefore not exact, in contrast to  $d\Phi_q$ .

We must still couple the harmonic gauge connection  $h$  with its nontrivial holonomies to our bosonic system. The interaction term in the action reads

$$S_{\Psi A \Psi} = -\frac{i}{2\pi} \int *J \wedge A = -\frac{i}{2\pi} \sum_{i=1}^{g+p-1} \left[ \oint_{\tilde{\alpha}_i} *J \oint_{\tilde{b}_i} A - \oint_{\tilde{b}_i} *J \oint_{\tilde{\alpha}_i} A \right], \quad (6.15)$$

with the generalized homology cycles  $\tilde{\alpha} = (\vec{a}, \vec{c})$  and  $\tilde{b} = (\vec{b}, \vec{d})$ . Inserting  $A = h$  and  $*J = id\Phi_s$  with winding numbers given in (5.13) and (6.11), respectively, we obtain

$$S_{\Psi A \Psi}(K, C) = 2\pi i \sum_{i=1}^{g+p-1} [\tilde{m}_i \tilde{B}_i - \tilde{n}_i \tilde{A}_i] = \pi [\bar{K}(\text{Im}T)^{-1} C - \bar{C}(\text{Im}T)^{-1} K], \quad (6.16)$$

the intersection number of the one-forms  $d\Phi_s$  and  $h$ . This implies that we must extend the bosonized action to

$$S'_\Phi(K, C) = \frac{\pi}{2} [\bar{K} - 2\bar{C}](\text{Im}T)^{-1} [K + 2C] + 2\pi \bar{C}(\text{Im}T)^{-1} C - \frac{1}{4\pi} \int d^2z \partial_{\bar{z}} \Phi_q \partial_z \Phi_q - c \Phi(P), \quad (6.17)$$

which is achieved by shifting

$$d\Phi_s \rightarrow d\Phi_s + 2 * h, \quad \text{that is} \quad K \rightarrow K + 2C \quad \text{but} \quad \bar{K} \rightarrow \bar{K} - 2\bar{C}. \quad (6.18)$$

Note that left- and right-movers are shifted in opposite ways since we have added a real  $*h$  to an imaginary  $d\Phi$ .<sup>7</sup> The general form of the bosonized action becomes

$$S'_\Phi = -\frac{1}{4\pi} \int \left[ \frac{1}{2} (d\Phi + 2 * h) \wedge (*d\Phi - 2h) + 2 * h \wedge h \right] - c \Phi(P) \quad (6.19)$$

where we dropped  $-c \int_{P_0}^P *h$  by choosing  $P_0 = P$ . Our recipe preserves the form of (6.13) and (6.11), but now with real winding numbers  $m_k, n_k, p_\ell, q_\ell \in \mathbb{R}$ . The combination of left- and right-moving fermionic correlators always yields just the bilinears

$$\frac{1}{2} \Psi_\downarrow^\pm \Psi_\uparrow^\pm = e^{\pm\phi \pm \bar{\phi}} = e^{\pm\Phi} \quad (6.20)$$

which are neither real nor  $U(1)$  neutral but have well-defined holonomies. From these the bosonized vertex operators are built.

In the bosonized path integral, we are instructed to sum over integral solitonic winding numbers for a fixed choice of spin structures, as well as to integrate over gauge moduli later on. The above prescription (6.18) combines the two. To make this more explicit, let us perform the standard soliton sum

$$f_s(C; D|T) = \sum_{K \in \frac{1}{2}\Lambda} \exp \left\{ -S'_\Phi[\Phi_s] + \sum_\ell \alpha_\ell \Phi_s(P_\ell) \right\} \quad (6.21)$$

where  $D = \sum_\ell \alpha_\ell P_\ell + cP$  is the divisor of degree  $c$  given by the vertex and Chern number insertions, and we use the same symbol for its image under the (generalized) Abel map,  $Z = \int_{P_0}^P \Omega$ . Shifting  $K$  according to (6.18), the exponent reads

$$\begin{aligned} & -S'_\Phi[\Phi_s] + \sum_\ell \alpha_\ell \Phi_s(P_\ell) \\ &= -\frac{\pi}{2} [\bar{K} - 2\bar{C}] \frac{1}{\text{Im}T} [K + 2C] - 2\pi\bar{C} \frac{1}{\text{Im}T} C + \pi [\bar{K} - 2\bar{C}] \frac{1}{\text{Im}T} D - \pi [K + 2C] \frac{1}{\text{Im}T} \bar{D} \\ &= -\frac{\pi}{2} [\bar{K} - 2\bar{C} + 2\bar{D}] \frac{1}{\text{Im}T} [K + 2C - 2D] - 2\pi\bar{C} \frac{1}{\text{Im}T} C - 2\pi\bar{D} \frac{1}{\text{Im}T} D. \end{aligned} \quad (6.22)$$

Splitting the sum into integral windings plus spin structures and applying the Poisson resummation formula [27, 28] yields

$$\begin{aligned} f_s(C; D|T) &= (\det \text{Im}T)^{1/2} e^{-2\pi \text{Re}D \frac{1}{\text{Im}T} \text{Re}D - 2\pi\bar{C} \frac{1}{\text{Im}T} C} \sum_S \left| \Theta[S](D - C|T) \right|^2 \\ &= (\det \text{Im}T)^{1/2} e^{\pi F(C, D)} \sum_S \left| \Theta[S - C](D|T) \right|^2, \end{aligned} \quad (6.23)$$

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<sup>7</sup> Again, the reason is that, in Euclidean space,  $S_\Psi$  is imaginary where  $S_\Phi$  is real.

where  $F(C, D)$  is some real function, and the sum runs over the (half-integral) spin structures,  $S \in (\mathbb{Z}/2\mathbb{Z})^{g+p-1} + T(\mathbb{Z}/2\mathbb{Z})^{g+p-1}$ . It is very satisfying that all the gauge moduli hide in the characteristics of the theta function. Since the characteristics of the theta function in a fermionic correlator represent the spin structure of the fermion system involved, the gauge moduli integral turns into an integral over continuous spin structures, respectively fermion holonomies, as it should. Indeed, the chiral fermionic correlator for an arbitrary spin structure  $C$  is proportional to  $\Theta[C](D|T)$ . Since  $S$  corresponds to half-points in the Jacobian torus, the integral over  $C$  is going to cover that torus  $2^{2(g+p-1)}$  times. For a single fermion system (as we have studied here), the final gauge moduli integral is simply

$$\int d^{2(g+p-1)} C f_s(C; D|T) = 2^{3(g+p-1)} \det(\text{Im}T) e^{-2\pi\bar{D}\frac{1}{\text{Im}T}D - 2\pi\bar{C}\frac{1}{\text{Im}T}C}, \quad (6.24)$$

as may be invoked directly by combining the sum and integral to a regular Gaussian integral. Due to spectral-flow invariance, the contribution from the punctures is actually trivial, as we shall make explicit in the following section. Their moduli may therefore be dropped altogether, restricting the puncture winding numbers  $p_\ell, q_\ell \in \mathbb{Z}$ . Thus, multi-loop computations simplify considerably as compared to the  $N=1$  string. In particular, there is no need for a GSO projection, since the sum over spin structures is implicit in the gauge moduli integration.

Finally, we will complete the bosonization of the  $N=2$  string by including the Lorentz index  $\mu$  and the ghost contributions. The prescription for bosonization of the left-moving charged  $N=2$  fields is given by (now dropping the tildes) [11]

$$\begin{aligned} \Psi^{\pm+} &= \Psi^{\pm 0} + \Psi^{\pm 1}, & \Psi^{\pm-} &= \Psi^{\pm 0} - \Psi^{\pm 1}, \\ \Psi^{\pm+}(z) \Psi^{\mp-}(w) &\sim \frac{-4}{z-w}, \\ \Psi^{\pm+} &= 2ie^{+\phi^\pm}, & \Psi^{\mp-} &= 2ie^{-\phi^\pm}, \\ \phi^\pm(z) \phi^\pm(w) &\sim +\log(z-w), \\ \gamma^\pm &= \eta^\pm e^{+\varphi^\pm}, & \beta^\mp &= e^{-\varphi^\pm} \partial \xi^\mp, \\ \delta(\gamma^\pm) &= e^{-\varphi^\pm}, & \delta(\beta^\mp) &= e^{+\varphi^\pm}, \\ \varphi^\pm(z) \varphi^\pm(w) &\sim -\log(z-w), & \xi^\mp(z) \eta^\pm(w) &\sim \frac{+1}{z-w}, \\ \eta^\pm &= e^{-\chi^\pm}, & \xi^\mp &= e^{\chi^\pm}, \\ \chi^\pm(z) \chi^\pm(w) &\sim +\log(z-w), \end{aligned} \quad (6.25)$$

and correspondingly for the right-movers. Applying our results obtained for a single complex fermion, we bosonize the relevant part of the gauge-fixed action and get

$$S_{U(1)} = -\frac{1}{4\pi} \int d^2z \left\{ \frac{1}{2} (\Psi_\downarrow^{-} \overleftrightarrow{\partial}_z \Psi_\downarrow^{++} + \Psi_\downarrow^{-+} \overleftrightarrow{\partial}_z \Psi_\downarrow^{+-}) - 4\beta^- \partial_z \gamma^+ - 4\beta^+ \partial_z \gamma^- \right.$$

$$\begin{aligned}
& +4iA_{\bar{z}}[-\frac{1}{4}(\Psi_{\downarrow}^{--}\Psi_{\downarrow}^{++} + \Psi_{\downarrow}^{-+}\Psi_{\downarrow}^{+-}) + \partial_{\bar{z}}(c\tilde{b}) + (\gamma^+\beta^- - \gamma^-\beta^+)] \\
& +c.c. \} \\
\sim & -\frac{1}{4\pi} \int d^2z \left\{ \partial_z \Phi^+ \partial_{\bar{z}} \Phi^+ + \partial_z \Phi^- \partial_{\bar{z}} \Phi^- - \partial_z \varphi^+ \partial_{\bar{z}} \varphi^+ - \partial_z \varphi^- \partial_{\bar{z}} \varphi^- \right. \\
& + \partial_z \chi^+ \partial_{\bar{z}} \chi^+ + \partial_z \chi^- \partial_{\bar{z}} \chi^- \\
& - 2iF_{z\bar{z}}[+\Phi^+ - \Phi^- - \varphi^+ + \varphi^- + c\tilde{b} - \bar{c}\tilde{b}] \\
& \left. + \sqrt{g}\mathcal{R}[\varphi^+ + \varphi^- - \frac{1}{2}(\chi^+ + \chi^-)] \right\}, \tag{6.26}
\end{aligned}$$

where again we retained the gauge field but also the coupling to the worldsheet curvature. Decomposing  $\Phi$  and  $\varphi$  into a single-valued “quantum” part and a harmonic solitonic part and inserting the explicit form of the latter, we arrive at

$$\begin{aligned}
S_{U(1)} = & \frac{\pi}{2}[\bar{K}^+(\text{Im}T)^{-1}K^+ + \bar{K}^-(\text{Im}T)^{-1}K^- - \bar{\mathcal{K}}^+(\text{Im}T)^{-1}\mathcal{K}^+ - \bar{\mathcal{K}}^-(\text{Im}T)^{-1}\mathcal{K}^-] \\
& - c[\Phi^+ - \Phi^- - \varphi^+ + \varphi^- + c\tilde{b} - \bar{c}\tilde{b}](P) - 4(g-1)[\varphi^+ + \varphi^- - \frac{1}{2}(\chi^+ + \chi^-)](P') \\
& - \frac{1}{4\pi} \int d^2z (\partial_z \Phi_q^+ \partial_{\bar{z}} \Phi_q^+ + \partial_z \Phi_q^- \partial_{\bar{z}} \Phi_q^- - \partial_z \varphi_q^+ \partial_{\bar{z}} \varphi_q^+ - \partial_z \varphi_q^- \partial_{\bar{z}} \varphi_q^-) \\
& - \frac{1}{4\pi} \int d^2z (\partial_z \chi^+ \partial_{\bar{z}} \chi^+ + \partial_z \chi^- \partial_{\bar{z}} \chi^-), \tag{6.27}
\end{aligned}$$

where  $K^{\pm}$  and  $\mathcal{K}^{\pm}$  contain the winding numbers of the solitons  $\Phi_s^{\pm}$  and  $\varphi_s^{\pm}$ , respectively. The  $U(1)$  field strength has been  $\delta$ -distributed at  $P$ , and the worldsheet curvature has been concentrated at  $P'$ .

Due to the “wrong” kinetic energy sign of the ghost bosons in (6.27), their soliton sums appear to diverge. This phenomenon is well known to also occur in the  $N=1$  string and originates in the overcounting of the naive bosonized  $(\beta, \gamma)$  path integral due to the picture degeneracy. Indeed, since two bosons ( $\varphi$  and  $\eta$ ) were used to bosonize each  $(\beta, \gamma)$  system, it appears that the solitonic winding lattice has doubled in dimension. The remedy [29] consists of inserting  $g$  projection operators, which contain a factor of  $|\oint \frac{dz}{2\pi i} \eta(z)|^2$  and fix the windings of  $\varphi - \chi$  to some chosen picture numbers. The puncture moduli are not affected since we will choose our vertex operators in definite pictures. The final result of the amplitude does not depend on those picture numbers.

Now that we have to deal with four different compact bosons, the gauge moduli integral is no longer as simple, because the four solitons share the fractional part  $C$  but not the integral part  $K$  of their windings. To make sure that all fermions have the same spin structure, however, one has to modify the bosonized description slightly, as is also done in the  $(1,1)$  string (prior to GSO projection). Instead of summing the theta-squared of each fermion over the half-integral spin-structures deriving from its soliton sum, we first multiply the theta-squares and then sum over common half-integral spin-structures. The latter sum again combines naturally with the gauge moduli integral.

The Chern number term in (6.27) contains exactly the bosonized form of the ghost-extended gauge current  $\hat{J}$ , integrated up to  $P$ . Abbreviating

$$\hat{\phi} = \phi^+ - \phi^- - \varphi^+ + \varphi^- + c\tilde{b}, \tag{6.28}$$

the weight factor  $e^{-S_{g,c}^{fix}}$  in (5.5) is related to the one at  $c = 0$  via

$$e^{-S_{g,c}^{fix}} = e^{-S_{g,0}^{fix}} e^{c\hat{\Phi}(P)} = e^{-S_{g,0}^{fix}} SFO(c, P) \overline{SFO}(c, P), \quad (6.29)$$

with the spectral-flow operator  $SFO$  now defined as [12, 13]

$$SFO(\Theta, z) = \exp\left\{-\Theta \int^z dw \hat{J}_w(w)\right\} = \exp\left\{\Theta \int^z dw \partial_w \hat{\phi}(w)\right\} = \exp(\Theta \hat{\phi}(z)), \quad (6.30)$$

and  $\overline{SFO}(\Theta) = SFO(-\Theta)^*$ . With  $z(P) = 0$ , it should be equivalent to the operator  $SFO(\Theta)$  defined in (4.6). The solitonic piece,  $e^{\Theta \hat{\Phi}_s(P)}$ , serves to shift the holonomies around  $P$ . The path-integral insertion of  $e^{c\hat{\Phi}(P)}$  originating from (6.29) creates a ( $c$ -fold) instanton where the  $U(1)$  field strength becomes singular and turns a  $c = 0$  amplitude into one at nonzero value of  $c$ . In the fermionic language, it arises from the coupling of the gauge current  $\hat{J}$  to the background gauge connection  $A_c$ , e.g. (5.24) and (5.25). Its presence changes not only the fermion zero mode count but also shifts the total sum of bosonic holonomies from zero to  $c$ . Since in the first-quantized formulation there is no a priori way to compare the normalizations of amplitudes with different values of  $c$  (or  $g$ ), the ambiguity hiding in the lower bound of the above integrals (for  $c \neq 0$ ) should be absorbed into a new coupling factor of  $\lambda^{-c}$ , consistent with amplitude factorization.

## 7 Vertex Operators and BRST Cohomology

The last missing ingredient to calculate string amplitudes is a set of vertex operators. A rather thorough (but not complete) investigation of their construction can be found in Ref.[11]. To obtain string vertex operators in explicit form, bosonization is very helpful. The bosonization prescription (6.25) for the susy ghosts introduced, in addition to  $\varphi^\pm$ , a system of auxiliary fermions  $\eta^\pm$  and  $\xi^\pm$ , which can also be bosonized by  $\chi^\pm$ . The bosonic ( $\varphi^\pm, \chi^\pm$ ) Fock space is much larger than the ( $\beta^\mp, \gamma^\pm$ ) Fock space, since the latter is fully represented once for each value of the picture numbers  $\pi^\pm$ , as measured by

$$\Pi^\pm = \oint [-\partial_z \varphi^\pm + \partial_z \chi^\pm]. \quad (7.1)$$

The known procedure to get from one picture to another makes use of the two picture-changing operators [25, 11]

$$PCO^\pm(z) = \{Q, \xi^\pm(z)\}. \quad (7.2)$$

Inserting the bosonized BRST-current

$$\begin{aligned} J_{BRST} = & cT + \eta^+ e^{+\varphi^+} G^- + \eta^- e^{+\varphi^-} G^+ + \tilde{c}(\partial\Phi^- - \partial\Phi^+) \\ & + c[\partial c b + \partial \tilde{c} \tilde{b} - \frac{1}{2}(\partial\varphi^+)^2 - \frac{1}{4}\partial^2\varphi^+ - \frac{1}{2}(\partial\varphi^-)^2 - \frac{1}{4}\partial^2\varphi^- - \eta^+ \partial\xi^- - \eta^- \partial\xi^+] \\ & - 4\eta^+ e^{\varphi^+} \eta^- e^{\varphi^-} \tilde{b} + 2\partial(\eta^- e^{\varphi^-}) \eta^+ e^{\varphi^+} \tilde{b} - 2\partial(\eta^+ e^{\varphi^+}) \eta^- e^{\varphi^-} \tilde{b} \\ & + \frac{3}{4}\partial c(\partial\varphi^+ + \partial\varphi^-) + \tilde{c}(\partial\varphi^+ - \partial\varphi^-), \end{aligned} \quad (7.3)$$

we find

$$PCO^\pm = c\partial\xi^\pm + e^{+\varphi^\mp}(G^\pm - 4\gamma^\pm b \mp 4\partial\gamma^\pm\tilde{b} \mp 2\gamma^\pm\partial\tilde{b}). \quad (7.4)$$

Since  $PCO^\pm$  are obviously BRST-closed, they map physical states to physical states. Furthermore,

$$PCO^+(z) PCO^+(w) \sim \text{regular}, \quad PCO^+(z) PCO^-(w) \sim \{Q, \text{singular}\} + \text{regular}, \quad (7.5)$$

so that picture-changing operators commute on physical states. Because  $\partial_z PCO^\pm$  is BRST-exact, picture-changing operators can be moved at will on the worldsheet, without changing the correlators of BRST-invariant operators. In contrast to the  $N=1$  string, however,  $PCO^\pm$  in (7.4) are *not locally invertible* and therefore do not immediately lead to the usual picture degeneracy of the BRST cohomology.

The spectral-flow operator encountered in the previous section provides us with another tool to link different pictures. It is BRST-closed since

$$[Q, SFO(\Theta, z)] = -\Theta\left[Q, \int^z \tilde{J}\right] SFO(\Theta, z) = -\Theta\left[Q, \int^z \{Q, \tilde{b}\}\right] SFO(\Theta, z) = 0 \quad (7.6)$$

but not BRST-exact, due to the one in the expansion of the exponential. In contrast, its derivative is BRST-exact,

$$\partial_z SFO(\Theta, z) = -\Theta J(z) SFO(\Theta, z) = -\Theta\{Q, \tilde{b}(z)\} SFO(\Theta, z) = -\Theta\{Q, \tilde{b}(z) SFO(\Theta, z)\}. \quad (7.7)$$

Thus,  $SFO$  may also be moved around in physical correlators. The spectral-flow invariance of physical correlators renders the gauge puncture moduli integration rather trivial. For each value of  $(\vec{G}, \vec{D})$ , a puncture twist may be collected in  $SFO(\Theta)\overline{SFO}(\bar{\Theta})$  and moved away, finally to cancel. Thus, it suffices to take all vertex operators in the NS sector and multiply the correlator with the (unit) volume of the gauge puncture moduli space. In contrast to  $PCO^\pm$ , spectral-flow operators are invertible by  $SFO(\Theta, z)^{-1} = SFO(-\Theta, z)$ , and form an abelian algebra via  $SFO(\Theta_1)SFO(\Theta_2) = SFO(\Theta_1 + \Theta_2)$ . Hence, we may insert

$$1 = SFO(\Sigma_i \Theta_i, z) = \prod_i SFO(\Theta_i, z) = \prod_i SFO(\Theta_i, z_i) + \{Q, *\} \quad \text{with} \quad \sum_i \Theta_i = 0 \quad (7.8)$$

into any physical correlator. Moreover, since  $SFO$  clearly commutes with  $PCO^\pm$  and changes the picture numbers by

$$(\pi^+, \pi^-) \longrightarrow (\pi^+ + \Theta, \pi^- - \Theta), \quad (7.9)$$

it realizes the one-to-one map of spectral flow between physical states in all pictures with level  $\pi := \pi^+ + \pi^- = \text{constant}$ . In particular, NS states at  $\pi^\pm \in \mathbb{Z}$  get related to R states at  $\pi^\pm \in \mathbb{Z} + \frac{1}{2}$ . Nevertheless, the BRST cohomology does change with  $\pi$ , since there are two *inequivalent* ways to move by  $\delta\pi^\pm = \frac{1}{2}$ ,

$$PCO^+ SFO(-\tfrac{1}{2}) - PCO^- SFO(+\tfrac{1}{2}) \neq \{Q, \text{something}\}. \quad (7.10)$$

To complete the picture, we briefly present the results of our earlier investigations of the BRST cohomology [11, 14]. Let us first impose the auxiliary conditions

$$b_0 = 0 = \bar{b}_0 \quad \text{and} \quad \tilde{c}_0 = 0 = \bar{\tilde{c}}_0 \quad (7.11)$$

to get rid of the trivial ghost zero mode degeneracy. The relative BRST cohomology then factorizes into identical left- and right-moving parts. It turns out that the left-moving BRST cohomology in any picture  $(\pi^+, \pi^-)$  is non-zero only at the massless level and for the left total ghost number  $u = \pi + 2$ .<sup>8</sup> There, however, one finds  $2j + 1 := \pi + 3$  different physical states [14]

$$|\pi^+, \pi^-, q\rangle = V_{(\pi^+, \pi^-)}^{(q)} |0\rangle, \quad q - (\pi^+ - \pi^-)/2 = -j, \dots, +j, \quad (7.12)$$

which are created from the  $SL(2)$  invariant neutral NS vacuum  $|0\rangle$  by vertex operators  $V$ , whose external momentum  $k$  we suppressed. Here,  $q$  denotes the charge under the  $U(1)$  factor of the global  $U(1, 1)$  Lorentz group. Since  $PCO^\pm$  are Lorentz singlets but  $SFO(\Theta)$  carries charge  $q = \Theta$ , the proliferation of physical states in higher pictures goes back to (7.10) whose two terms have different  $q$  charge. This leads to inequivalent words (different  $q$ ) of the same “length”  $\pi$  built from  $PCO^\pm$  ( $\pi=1$ ) and  $SFO$  ( $\pi=0$ ) who, acting on  $|-1, -1, 0\rangle$ , create inequivalent physical states at picture  $(\pi^+, \pi^-)$ .

Thus, the number of physical states seems to grow linearly with the picture number, but this is not so. In fact, it can be shown [14] that the states in each  $(2j + 1)$ -plet are proportional to each other in a nonlocal way, i.e. involving inverse powers of external momenta. Combining left- and right-movers again, this multiplicity is easily understood from the space-time point of view.<sup>9</sup> In the lowest picture,  $\pi = \bar{\pi} = -2$  hence  $j = \bar{j} = 0$ , one has a single scalar state with massless momentum  $k$ , which corresponds to a massless scalar field  $K(Z^{\pm\mu})$ . The space-time effect of applying  $PCO^\pm$  or  $\overline{PCO}^\pm$  is to create space-time derivatives,  $\partial_{\pm\mu} K$ , at level  $\pi + \bar{\pi} = -3$ , represented by the two doublets  $(j = \frac{1}{2}, \bar{j} = 0)$  and  $(j = 0, \bar{j} = \frac{1}{2})$ . In this way, any higher derivative of  $K$  is represented in some higher picture, but nothing more. Furthermore, equating  $PCO$  with space-time derivatives “explains” why picture-changing is not (locally) invertible in the  $N=2$  string. Interestingly, the instanton-creation operator,  $SFO(+1)$ , also has a space-time interpretation, but as one of the missing boost generators which complete the  $SO(2, 2)$  Lorentz algebra [14].

The canonical vertex operator to be used in our calculational scheme for computing string amplitudes (see (5.5)) is

$$V_{(-1, -1)}^{(0)} \bar{V}_{(-1, -1)}^{(0)}(k) = \tilde{c} c e^{-\varphi^- - \varphi^+} \bar{\tilde{c}} \bar{c} e^{-\bar{\varphi}^- - \bar{\varphi}^+} e^{ik \cdot Z}, \quad (7.13)$$

with  $k \cdot Z = \frac{1}{2}(k^{+\mu} Z^-_{\mu} + k^{-\mu} Z^+_{\mu})$ . BRST invariance then requires  $k^2 = 0$ . As explained above, any vertex operator in a higher picture can be constructed by applying an appropriate  $PCO/SFO$  word to (7.13). In fact, such vertex operators appear when moving some

<sup>8</sup> This has yet to be proven for all higher pictures.

<sup>9</sup> O.L. thanks A. Morozov for this suggestion.



of the picture-changing insertions in (5.5) and the  $SFO(c)$  insertion (derived at the end of the previous section) onto punctures. More generally, using the invariance of physical correlators under picture-changing and spectral flow (7.8), we may assign any picture and  $q$ -charge labels to the vertex operators  $V_\ell$  in a correlator, as long as the selection rules

$$\sum_{\ell} \pi_{\ell}^{\pm} = 2g - 2 \quad \text{and} \quad \sum_{\ell} \theta_{\ell} = c \quad (7.14)$$

are obeyed (using the action  $S_{0,0}^{fix}$ ). Furthermore, those antighost insertions  $\langle \mu_i | b \rangle$  and  $\langle \omega_k | \tilde{b} \rangle$  associated with punctures may be contracted with their vertex operators. A  $b$  insertion will replace the  $c$  in (7.13) by an integral, and a  $\tilde{b}$  insertion removes the  $\tilde{c}$  in front of (7.13). A list of vertex operators for  $\pi^{\pm} = -1, -\frac{1}{2}, 0$  and  $q = 0, \pm\frac{1}{2}$  can be found in Ref.[11].

## 8 Conclusions

We have presented a detailed account of the road leading from the formal path integral to the actual computation of arbitrary scattering amplitudes, for the critical  $N=(2,2)$  string on a flat background. Using the NSR formulation and including external leg punctures from the beginning, this entailed a careful gauge-fixing and discussion of the extended  $(2,2)$  supermoduli space, with special emphasis on the gauge moduli arising from the  $U(1)$  gauge field of  $N=2$  worldsheet supergravity. The explicit realization of spectral flow and its relation to the gauge moduli was clarified in particular by employing bosonization in the Abelian gauge background. Here, solitonic bosons were allowed to carry any real winding number, and the resulting spin structure sum found its anticipated extension by the gauge moduli integral. Nontrivial gauge bundle topologies were constructed explicitly and could be reached through the creation of worldsheet Maxwell instantons by a local operator generalizing spectral flow. Finally, we summarized the BRST cohomology of physical states and vertex operators, explaining its novel subtleties due to picture changing and spectral flow. A new interpretation of picture-changing operators as space-time momenta explained their non-invertibility in the  $N=2$  string.

Having collected all ingredients to evaluate scattering amplitudes at any loop and instanton number, we stopped just short of that. For once, the tree-level 3- and 4-point functions have already been computed in this framework. Furthermore, the results depend on the choice of allowing for either a single (common left-right) Chern number  $c$  or else, two distinct (left and right) Chern numbers  $c_L$  and  $c_R$ . Since the latter case has been discussed in [8, 9, 14], let us close this work with some comments on the former case.

The first nontrivial example is that of the tree-level 3-point function, where  $|c| \leq 1$ . On the sphere, all gauge moduli are trivial and metric moduli are absent. The two picture-changing operator insertions are moved to two punctures and change their picture assignments, depending on  $c$ . For  $c=1$ , the instanton-creation operator  $SFO(+1)$  is moved to one of the vertex operators, increasing its  $q$  charge and converting the picture-assignments to those in the  $c=0$  sector. Likewise for  $c=-1$ . The relevant chiral correlators

may thus be computed as [13, 14]

$$\begin{aligned}
\tilde{A}_{0,0}^3 &= \left\langle V_{(-1,-1)}^{(0)}(k_1) \oint \tilde{b} V_{(-1,-1)}^{(0)}(k_2) \oint \tilde{b} V_{(-1,-1)}^{(0)}(k_3) PCO^- PCO^+ \right\rangle \\
&= \left\langle V_{(-1,-1)}^{(0)}(k_1) \oint \tilde{b} V_{(-1,0)}^{(0)}(k_2) \oint \tilde{b} V_{(0,-1)}^{(0)}(k_3) \right\rangle = -\frac{1}{2}(k_2^+ \cdot k_3^- - k_2^- \cdot k_3^+) \\
\tilde{A}_{0,+1}^3 &= \left\langle V_{(-1,-1)}^{(0)}(k_1) \oint \tilde{b} V_{(-1,-1)}^{(0)}(k_2) \oint \tilde{b} V_{(-1,-1)}^{(0)}(k_3) PCO^- PCO^- SFO(+1) \right\rangle \\
&= \left\langle V_{(-1,-1)}^{(0)}(k_1) \oint \tilde{b} V_{(-1,0)}^{(0)}(k_2) \oint \tilde{b} V_{(0,-1)}^{(+1)}(k_3) \right\rangle = -k_2^+ \wedge k_3^+ \\
\tilde{A}_{0,-1}^3 &= \left\langle V_{(-1,-1)}^{(0)}(k_1) \oint \tilde{b} V_{(-1,-1)}^{(0)}(k_2) \oint \tilde{b} V_{(-1,-1)}^{(0)}(k_3) PCO^+ PCO^+ SFO(-1) \right\rangle \\
&= \left\langle V_{(-1,-1)}^{(0)}(k_1) \oint \tilde{b} V_{(-1,0)}^{(-1)}(k_2) \oint \tilde{b} V_{(0,-1)}^{(0)}(k_3) \right\rangle = +k_2^- \wedge k_3^-, \tag{8.1}
\end{aligned}$$

where  $\langle \dots \rangle$  denotes the averaging with the action  $S_{0,0}^{fix}$ . According to (6.29), left- and right-movers are to be combined using the same values for  $c$ , so that the total closed-string tree-level amplitude (including Maxwell instantons) reads

$$A_0^3 = [\tilde{A}_{0,0}^3]^2 + \lambda^{-2}[\tilde{A}_{0,+1}^3]^2 + \lambda^{+2}[\tilde{A}_{0,-1}^3]^2. \tag{8.2}$$

This result seems not to be real or neutral under the  $U(1)$  factor of the Lorentz group, but this impression is faulty. Taking into account the origin (6.30) of the Maxwell coupling  $\lambda$ , together with the fact that  $SFO(\Theta)$  carries Lorentz charge  $q = \Theta$ , one must consider  $\lambda$  as a phase and also assign it a charge  $q = 1$ . Obviously, this restores the reality and  $U(1, 1)$  invariance of  $A_0^3$ . One should note, however, that the amplitude (8.2) is rather different from the result of the  $c_L \neq c_R$  construction [14],

$$A_0'^3 = [2\tilde{A}_{0,0}^3 + \lambda^{-1}\tilde{A}_{0,+1}^3 + \lambda^{+1}\tilde{A}_{0,-1}^3]^2, \tag{8.3}$$

which offers the possibility of extending the Lorentz group to  $SO(2, 2)$ .

There remain a number of interesting unresolved issues. Among them are the structure of one- and higher-loop amplitudes, a simple proof of the vanishing of  $(n \geq 4)$ -point functions (at least at tree-level) and the extension of this work to open  $N=2$  string case, including the interaction of open with closed strings. The latter is particularly important in order to decide upon the correct coupling of self-dual Yang-Mills theory to self-dual gravity in four dimensions [30]. We intend to report on these issues in the not too distant future.

## Acknowledgement

O.L. acknowledges enlightening discussions with Gordon Chalmers, Michael Flohr, Amihay Hanany, Sergei Ketov, Andrei Losev, Greg Moore, Alexei Morozov, V.P. Nair, Warren Siegel and Samson Shatashvili. He is also grateful to the Institute for Advanced Study for hospitality during the final stages of this work. J.B. would like to thank Peter Adamietz and Klaus Jünemann for fruitful discussions.

# A Conventions

Spinor Notations:

$$\Psi^{\pm\mu} = \Psi^{2\mu} \pm i\Psi^{3\mu}, \quad \Psi^{\pm\mu} = \begin{pmatrix} \Psi_{\uparrow}^{\pm\mu} \\ \Psi_{\downarrow}^{\pm\mu} \end{pmatrix}, \quad \bar{\Psi} = \Psi^{\dagger}\gamma^0. \quad (\text{A.1})$$

with  $\Psi^{i\mu}; i = 2, 3; \mu = 0, 1$  Majorana-Spinors.  $\mu$  is the  $SO(1, 1)$  Lorentz index and  $i$  the  $SO(2)$  Euclidean index of the total  $SO(2, 2)$  Lorentz group. Dirac algebra:

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.2})$$

$$\begin{aligned} \{\gamma^a, \gamma^b\} &= 2\eta^{ab}, & \gamma^a\gamma^b &= \eta^{ab} + \epsilon^{ab}\gamma_5, \\ \gamma_b\gamma^a\gamma^b &= 0, & \gamma^{\mu\dagger} &= \gamma^0\gamma^{\mu}\gamma^0 \\ \eta^{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \epsilon^{01} &= 1 = \epsilon_{10}, \end{aligned} \quad (\text{A.3})$$

Lorentz algebra and its representations:

$$[L^{ab}, L^{cd}] = \eta^{ad}L^{bc} - \eta^{bd}L^{ac} - \eta^{bc}L^{ad} + \eta^{ac}L^{bd}. \quad (\text{A.4})$$

$$\begin{aligned} (L^{ab})_r{}^s v_s &= (\delta_r^a \eta^{bs} - \delta_r^b \eta^{as}) v_s, \\ L^{ab}\Psi &= \frac{1}{4} [\gamma^a, \gamma^b] \Psi = \frac{1}{2} \epsilon^{ab} \gamma_5 \Psi. \end{aligned} \quad (\text{A.5})$$

Lorentz covariant derivatives:

$$\begin{aligned} D_m v^a &= \partial_m v^a + \frac{1}{2} \omega_{mcd} (L^{cd})_b{}^a v^b = \partial_m v^a + \omega_m \epsilon^a{}_b v^b, \\ D_m \Psi &= \partial_m \Psi + \frac{1}{2} \omega_{mcd} (L^{cd}) \Psi = \partial_m \Psi + \frac{1}{2} \omega_m \gamma_5 \Psi, \end{aligned} \quad (\text{A.6})$$

with

$$\omega_m = \frac{1}{2} \omega_{mab} \epsilon^{ab}, \quad e_a{}^m \omega_m = \epsilon^{nl} (\partial_l e_{na} + i \bar{\chi}_l^- \gamma_a \chi_n^+). \quad (\text{A.7})$$

Light-cone coordinates on the worldsheet:

$$\begin{aligned} x^{\pm} &= x^0 \pm x^1, & \partial_{\pm} &= \frac{1}{2} (\partial_0 \pm \partial_1), \\ \eta^{+-} &= 2, & \eta_{-+} &= \frac{1}{2}, \end{aligned} \quad (\text{A.8})$$

$$\gamma^+ = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.9})$$

Wick-rotated ( $x^0 \rightarrow -ix^0$ ) coordinates and gamma matrices:

$$\begin{aligned} x^+ &\rightarrow z, & x^- &\rightarrow \bar{z}, & \partial_+ &\rightarrow \partial_z, & \partial_- &\rightarrow \partial_{\bar{z}}, \\ \gamma^0 &\rightarrow -i\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \gamma^z &= \gamma^+, & \gamma^{\bar{z}} &= \gamma^-. \end{aligned} \quad (\text{A.10})$$

Covariant derivatives in complex coordinates:

$$\begin{aligned} D_z \chi_{\bar{z}\uparrow}^\pm &= \partial_z \chi_{\bar{z}\uparrow}^\pm - \frac{1}{2} \omega_z \chi_{\bar{z}\uparrow}^\pm, & D_{\bar{z}} \chi_{z\downarrow}^\pm &= \partial_{\bar{z}} \chi_{z\downarrow}^\pm + \frac{1}{2} \omega_{\bar{z}} \chi_{z\downarrow}^\pm, \\ D_z \tilde{\epsilon}_\uparrow^\pm &= \partial_z \tilde{\epsilon}_\uparrow^\pm + \frac{1}{2} \omega_z \tilde{\epsilon}_\uparrow^\pm, & D_{\bar{z}} \tilde{\epsilon}_\downarrow^\pm &= \partial_{\bar{z}} \tilde{\epsilon}_\downarrow^\pm - \frac{1}{2} \omega_{\bar{z}} \tilde{\epsilon}_\downarrow^\pm, \\ D_z \xi^{\bar{z}} &= \partial_z \xi^{\bar{z}} + \omega_z \xi^{\bar{z}}, & D_{\bar{z}} \xi^z &= \partial_{\bar{z}} \xi^z - \omega_{\bar{z}} \xi^z. \end{aligned} \quad (\text{A.11})$$

$$\mathcal{D}_m \epsilon^\pm = (D_m \mp iA_m) \epsilon^\pm, \quad \mathcal{D}_m \chi_n^\pm = (D_m \mp iA_m) \chi_n^\pm \quad (\text{A.12})$$

One- and two-forms in local complex coordinates:

$$\begin{aligned} \omega &= \omega_z dz + \omega_{\bar{z}} d\bar{z}, \\ df &= \partial_z f dz + \partial_{\bar{z}} f d\bar{z}, \\ \phi &= \phi_{z\bar{z}} dz \wedge d\bar{z}, \\ d\omega &= (\partial_z \omega_{\bar{z}} - \partial_{\bar{z}} \omega_z) dz \wedge d\bar{z}. \end{aligned} \quad (\text{A.13})$$

Hodge star operator:

$$\begin{aligned} * \omega &= -i\omega_z dz + i\omega_{\bar{z}} d\bar{z}, \\ * \phi &= -ig^{z\bar{z}} \phi_{z\bar{z}}. \end{aligned} \quad (\text{A.14})$$

Normalized canonical real one-forms:

$$\begin{aligned} \oint_{a_i} \alpha_j &= \oint_{b_i} \beta_j = \delta_{ij}, \\ \oint_{b_i} \alpha_j &= \oint_{a_i} \beta_j = 0. \end{aligned} \quad (\text{A.15})$$

Relation with holomorphic one-forms:

$$\begin{aligned} \omega_i &= \alpha_i + \tau_{ij} \beta_j, & \bar{\omega}_i &= \alpha_i + \bar{\tau}_{ij} \beta_j, \\ \oint_{a_i} \omega_j &= \delta_{ij}, & \oint_{b_i} \omega_j &= \tau_{ij}, \\ \tau_{ij} &= \tau_{ji}, & \text{Im} \tau_{ij} &> 0, \\ \beta_i &= (\tau - \bar{\tau})_{ij}^{-1} (\omega_j - \bar{\omega}_j), \\ \alpha_i &= (\bar{\tau}(\bar{\tau} - \tau)^{-1})_{ij} \omega_j + (\tau(\tau - \bar{\tau})^{-1})_{ij} \bar{\omega}_j. \end{aligned} \quad (\text{A.16})$$

Volume forms in complex coordinates  $z = x + iy$ :

$$\begin{aligned} dz \wedge d\bar{z} &= -id^2z = -2idx \wedge dy = -2id^2x \\ \Omega &= \frac{i}{2g} \omega_i (\text{Im}\tau)_{ij}^{-1} \bar{\omega}_j \end{aligned} \quad (\text{A.17})$$

Greens and delta functions:

$$\begin{aligned} \int d^2x \delta^2(\vec{x}) &= 1 = \int d^2z \delta^2(z), \\ \delta^2(\vec{x}) &= 2\delta^2(z), \\ \partial_x^2 \partial_y^2 \ln(r) &= 2\pi \delta^2(\vec{x}) = 2\partial_z \partial_{\bar{z}} \ln(z\bar{z}) = 4\pi \delta^2(z). \end{aligned} \quad (\text{A.18})$$

$$\partial_{\bar{z}} 1/z = \partial_z 1/\bar{z} = 2\pi \delta^2(z). \quad (\text{A.19})$$

Integration of two closed one-forms:

$$\int_{\mathcal{M}} \Theta \wedge \tilde{\Theta} = \sum_{k=1}^g [\oint_{a_k} \Theta \oint_{b_k} \tilde{\Theta} - \oint_{b_k} \Theta \oint_{a_k} \tilde{\Theta}]. \quad (\text{A.20})$$

Lorentz and gauge covariant derivatives (with  $n(T^z) = 1$ ):

$$\begin{aligned} \mathcal{D}_z^{n,q} &= \partial_z + n\omega_{zz} z + 2iqA_z = \partial_z + n\omega_z \varepsilon_z^z + 2iqA_z \\ &= \partial_z - n\omega_z + 2iqA_z, \\ \mathcal{D}_{\bar{z}}^{n,q} &= \partial_{\bar{z}} - n\omega_{\bar{z}} + 2iqA_{\bar{z}}. \end{aligned} \quad (\text{A.21})$$

Lorentz covariant derivatives:  $D^n = \mathcal{D}^{n,o}$

$$\partial_m(\sqrt{g}v^m) = \sqrt{g}D_m^1 v^m \quad \implies \quad \partial_z(\sqrt{g}v^z) = \sqrt{g}(\partial_z - \omega_z)v^z, \quad (\text{A.22})$$

for a superconformal gauge with  $\chi_{\bar{z}\downarrow}^\pm, \chi_{z\uparrow}^\pm = 0$ . Also,  $\overline{\omega_z} = -\omega_{\bar{z}}$ .

Adjoint derivatives:

$$\begin{aligned} \int d^2z \sqrt{g} [\bar{\Psi}^\lambda \mathcal{D}_z^{1-\lambda,q} \Psi^{1-\lambda}] &= \int d^2z \sqrt{g} [\bar{\Psi}^\lambda (\partial_z - (1-\lambda)\omega_z + 2iqA_z) \Psi^{1-\lambda}] \\ &= - \int d^2z \sqrt{g} [(\partial_{\bar{z}} + \lambda\omega_{\bar{z}} + 2iqA_{\bar{z}}) \bar{\Psi}^\lambda \Psi^{1-\lambda}] \\ &= - \int d^2z \sqrt{g} [\overline{\mathcal{D}_{\bar{z}}^{-\lambda,q} \Psi^\lambda} \Psi^{1-\lambda}], \end{aligned} \quad (\text{A.23})$$

$$(\mathcal{D}_z^{n,q})^\dagger = -\mathcal{D}_{\bar{z}}^{n-1,q}, \quad (\mathcal{D}_{\bar{z}}^{n,q})^\dagger = -\mathcal{D}_z^{n-1,q}. \quad (\text{A.24})$$

Index theorem:

$$\begin{aligned} \dim \ker(\mathcal{D}_z^{n,q}) - \dim \ker(\mathcal{D}_z^{n,q})^\dagger &= (2n-1)(g-1) + 2qc, \\ \dim \ker(\mathcal{D}_{\bar{z}}^{n,q})^\dagger - \dim \ker(\mathcal{D}_{\bar{z}}^{n,q}) &= (2n+1)(g-1) + 2qc. \end{aligned} \quad (\text{A.25})$$

## B Spectral Flow Operator

On the modes of the symmetry generators the *SFO* has to act by

$$\begin{aligned} SFO(\Theta)L_nSFO(\Theta)^{-1} &= L_n + \Theta J_n + \Theta^2 \frac{c}{6} \delta_n, \\ SFO(\Theta)J_nSFO(\Theta)^{-1} &= J_n + \Theta \frac{c}{3} \delta_n, \\ SFO(\Theta)G_r^\pm SFO(\Theta)^{-1} &= G_{r \mp \Theta}^\pm. \end{aligned} \quad (\text{B.1})$$

We make an ansatz for a possible *SFO* of the following form

$$SFO(\Theta) = e^{\Theta\Omega} \quad \rightarrow \quad SFO(\Theta)^{-1} = e^{-\Theta\Omega}. \quad (\text{B.2})$$

Let us start with the spectral flow for the fermionic currents

$$SFO(\Theta)G_r^\pm SFO(\Theta)^{-1} = G_r^\pm + \Theta[\Omega, G_r^\pm] + \frac{\Theta^2}{2}[\Omega, [\Omega, G_r^\pm]] + \dots \quad (\text{B.3})$$

Considering the power expansion of the fermionic currents for an infinitesimal spectral flow with  $\Theta \ll 1$ , we obtain

$$\begin{aligned} G_{r \mp \Theta}^\pm &= \oint \frac{dz}{2\pi i} z^{r \mp \Theta + 1/2} G^\pm(z) = \oint \frac{dz}{2\pi i} z^{r+1/2} (1 \mp \Theta \log(z)) G^\pm(z) + \dots \\ &= G_r^\pm \mp \Theta \sum_n \oint \frac{dz}{2\pi i} z^{-n-1} \log(z) G_{r+n}^\pm = G_r^\pm \mp \Theta \sum_n c_n G_{r+n}^\pm \\ &= G_r^\pm + \Theta \sum_n c_n [J_n, G_r^\pm] = G_r^\pm + \Theta[\Omega, G_r^\pm] \end{aligned} \quad (\text{B.4})$$

with

$$c_n = \oint \frac{dz}{2\pi i} z^{-n-1} \log(z), \quad \Omega = \oint \frac{dz}{2\pi i} \log(z) J(z). \quad (\text{B.5})$$

We get a first hint for the exponent  $\Omega$  in (B.2). In order to induce the spectral flow on the  $L$ - and  $J$ -modes as well, our operator  $\Omega$  has to satisfy the following equations

$$[\Omega, L_m] = J_m, \quad [\Omega, J_m] = \delta_m. \quad (\text{B.6})$$

This is only possible if the weird identity

$$c_n = \frac{1}{n} \delta_n \quad (\text{B.7})$$

is valid. A rather formal calculation gives indeed

$$c_n = \oint \frac{dz}{2\pi i} z^{-n-1} \log(z) = - \oint \frac{dz}{2\pi i} \frac{1}{n} \partial_z (z^{-n}) \log(z) = \oint \frac{dz}{2\pi i} \frac{1}{n} z^{-n-1} = \frac{1}{n} \delta_n. \quad (\text{B.8})$$

Thus, the spectral flow operator given by

$$SFO(\Theta) = \exp(\Theta \oint \frac{dz}{2\pi i} \log(z) J(z)) \quad (\text{B.9})$$

acts on the modes as well as on the conformal fields in the right fashion.

## C Punctured Riemann Surfaces

Punctured Riemann surfaces enter the game at the moment one starts to consider  $p$ -string scattering amplitudes. The asymptotic incoming and outgoing string states can be conformally mapped to points on the compact Riemann surface representing the string Feynman diagram. The boundary conditions at these points give the quantum numbers of the asymptotic states, for example the momenta. These boundary conditions may be encoded by the insertion of vertex operators in the path integral over conformal fields living on a punctured Riemann surface [17]. Instead of treating punctured Riemann surfaces, it is usually more convenient to consider the well-known compact surfaces, endowed with meromorphic differentials which become singular at specified points. These two points of view are equivalent.

Let us first define a standard meromorphic one-form  $\omega_{PQ}$  having only simple poles at the points  $P$  and  $Q$  with the following properties

$$\begin{aligned} \text{res}_P \omega_{PQ} &= -\text{res}_Q \omega_{PQ} = \frac{1}{2\pi i}, \\ \oint_{a_i} \omega_{PQ} &= 0, \quad i = 1, \dots, g. \end{aligned} \quad (\text{C.1})$$

Such a choice of properties is always possible and determines  $\omega_{PQ}$  uniquely [26]. In fact all meromorphic differentials have to satisfy

$$\sum_{P \in \mathcal{M}} \text{res}_P m = 0. \quad (\text{C.2})$$

Every meromorphic one-form with only simple poles with

$$\begin{aligned} \text{res}_{P_\ell} m &= \theta_\ell \quad \text{for } P_\ell \in \mathcal{M}, \quad \ell = 1, \dots, p, & \sum_\ell \theta_\ell &= 0, \\ \text{res}_P m &= 0 \quad \text{for } P \in \mathcal{M} \setminus \{P_1, \dots, P_p\}, \\ \oint_{a_i} m &= 0 \quad \text{for } i = 1, \dots, g, \end{aligned} \quad (\text{C.3})$$

can be constructed with only the standard one-forms,

$$m = \sum_{\ell=1}^{p-1} 2\pi i \left( \sum_{k=1}^{\ell} \theta_k \right) \omega_{P_\ell P_{\ell+1}}. \quad (\text{C.4})$$

In this sense the one-forms  $\omega_{P_\ell P_{\ell+1}}$  for  $\ell = 1, \dots, p-1$  are extending the standard basis  $\{\omega_i, i = 1, \dots, g\}$  of holomorphic one-forms on a genus  $g$  Riemann surface to a basis for meromorphic one-forms with at most  $p$  simple poles. This basis is equivalent to a basis of holomorphic one-forms on a  $p$ -punctured surface.

We also have to extend the homology basis of the surface, which is dual to the basis of one-forms, by adding cycles that surround the points  $P_\ell$ . The appropriate choice for these cycles  $c_\ell$ ,  $\ell = 1, \dots, p-1$  is given in a way that  $c_\ell$  encloses the points  $P_1$  until  $P_\ell$ , see Figure 3. The line integrals of  $\omega_{P_\ell P_{\ell+1}}$  along these cycles then have the standard form

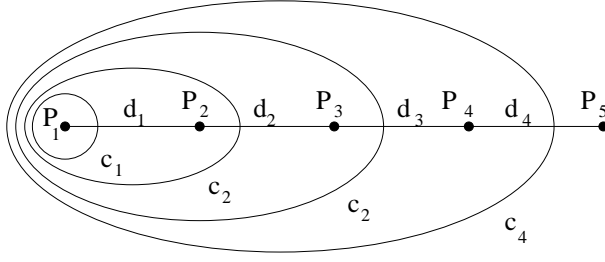


Figure 3: Additional cycles for a 5-punctured surface.

$$\oint_{c_k} \omega_{P_\ell P_{\ell+1}} = \delta_{\ell k}. \quad (\text{C.5})$$

Besides the cycles  $c_\ell$  we must also introduce the cycles  $d_\ell$  starting at  $P_\ell$  and ending at  $P_{\ell+1}$ .

All these structures on a  $p$ -punctured Riemann surface of genus  $g$  are identical to the structures of a degenerated compact genus  $g + p - 1$  surface which is obtained from the punctured surface by identifying all the points  $P_\ell$ . The cycles  $c_\ell$  and  $d_\ell$  then correspond to the additional  $p - 1$   $a$ - and  $b$ -cycles one gets for the degenerated compact surface.

A reasonable definition for the homology basis of a  $p$ -punctured surface is

$$\begin{aligned} \{a_1, \dots, a_g, c_1, \dots, c_{p-1}\} &\equiv \{a_1, \dots, a_{g+p-1}\}, \\ \{b_1, \dots, b_g, d_1, \dots, d_{p-1}\} &\equiv \{b_1, \dots, b_{g+p-1}\}. \end{aligned} \quad (\text{C.6})$$

Its dual basis is given by the meromorphic one-forms

$$\{\omega_1, \dots, \omega_g, \omega_{P_1 P_2}, \dots, \omega_{P_{p-1} P_p}\} \equiv \{\omega_1, \dots, \omega_{g+p-1}\} \quad (\text{C.7})$$

normalized to

$$\oint_{a_i} \omega_j = \delta_{ij} \quad \text{for } i, j = 1, \dots, g + p - 1. \quad (\text{C.8})$$

The generalized  $b$  cycle periods are ( $i, j = 1, \dots, g$  here)

$$\begin{aligned} \oint_{b_i} \omega_j &= \tau_{ij}, & \oint_{b_i} \omega_{P_\ell P_{\ell+1}} &= \rho_{i\ell}, \\ \oint_{d_\ell} \omega_i &= \bar{\rho}_{\ell i}, & \oint_{d_\ell} \omega_k &= \sigma_{\ell k}. \end{aligned} \quad (\text{C.9})$$

One consequently obtains a  $g + p - 1$ -dimensional period matrix  $T$

$$\oint_{b_i} \omega_j = T_{ij} = \begin{pmatrix} \tau & \rho \\ \bar{\rho} & \sigma \end{pmatrix}_{ij}, \quad (\text{C.10})$$

where  $i, j = 1, \dots, g + p - 1$  again. The degeneration of the Riemann surface by pinching  $p - 1$  handles to one point is reflected in the divergence of the diagonal  $\sigma$  periods

$$\sigma_{\ell\ell} = \int_{P_\ell}^{P_{\ell+1}} \omega_{P_\ell P_{\ell+1}} \quad (\text{C.11})$$



which blow up logarithmically at their endpoints.

We go on to construct bases for the meromorphic differentials of the weight 2,  $\frac{3}{2}$ , and 1, as needed to span the moduli spaces of the  $N=2$  string. Since a one-form  $\hat{\omega}$  with  $p$  simple poles and vanishing  $a$ -periods is uniquely determined for given residues it may also be uniquely constructed out of the differentials  $\omega_{P_\ell P_{\ell+1}}$ . The Riemann-Roch theorem demands that  $\deg(\hat{\omega}) = 2g-2$ , which implies that  $\hat{\omega}$  possesses  $2g-2+p$  simple zeroes, say at  $Q_0, \dots, Q_{2g-3+p}$ . With the help of  $\hat{\omega}$  we are able to construct the following meromorphic differentials [17]:

- $3g - 3 + p$  quadratic differentials:

$$\begin{aligned} \hat{\omega}\omega_i & \quad \text{for } i = 1, \dots, g, \\ \hat{\omega}\omega_{Q_0 Q_j} & \quad \text{for } j = 1, \dots, 2g - 3 + p. \end{aligned} \tag{C.12}$$

- $2g - 2 + p$  three-half differentials:

For even spin structure there exists no holomorphic but one meromorphic half-differential  $\kappa_Q$  with a single pole at  $Q$ , so one has

$$\hat{\omega}\kappa_{Q_j} \quad \text{for } j = 0, \dots, 2g - 3 + p. \tag{C.13}$$

For odd spin structure there exists one holomorphic half-differential  $h$  and in addition a meromorphic half-differential  $\kappa_{PQ}$  with single poles at  $P, Q$ , allowing for

$$\begin{aligned} \hat{\omega}\kappa_{Q_0 Q_j} & \quad \text{for } j = 1, \dots, 2g - 3 + p, \\ \hat{\omega}h & \end{aligned} \tag{C.14}$$

- $g - 1 + p$  one-forms:

these were already given in (C.7), namely

$$\omega_i \quad \text{for } i = 0, \dots, g - 1 + p. \tag{C.15}$$

Since punctured Riemann surfaces correspond to degenerated compact Riemann surfaces of a higher genus, it is possible to generalize the integration rules for compact surfaces and holomorphic, respectively antiholomorphic differentials  $\omega, \bar{\omega}$ , given by

$$\int_{\mathcal{M}} \omega \wedge \bar{\omega} = \sum_{i=1}^g [\oint_{a_i} \omega \oint_{b_i} \bar{\omega} - \oint_{b_i} \omega \oint_{a_i} \bar{\omega}].$$

Let us first consider only two punctures  $P, Q$  and the integral

$$\int \omega \wedge \bar{\omega}_{PQ}, \tag{C.16}$$

where  $\omega_{PQ}$  has the residues  $r$  and  $-r$  at  $P$  and  $Q$ , respectively. For the calculation of the integral above it is again convenient to cut the 2-punctured surface along its extended

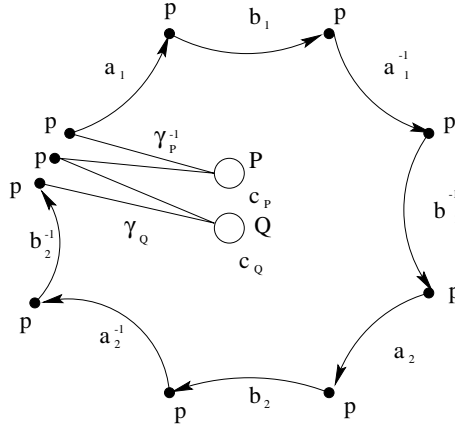


Figure 4: Cut 2-punctured Riemann surface.

homology basis, see figure 4. This cut Riemann surface  $\mathcal{M}'$  is simply connected, the differentials  $\omega, \bar{\omega}_{PQ}$  are (anti)holomorphic and thus closed on  $\mathcal{M}'$ . By the Poincaré lemma  $\omega, \bar{\omega}_{PQ}$  are also exact,

$$\omega = df, \quad f = \int_{z_0}^z \omega. \quad (\text{C.17})$$

Using the exactness in the integral, one obtains

$$\begin{aligned} \int_{\mathcal{M}'} \omega \wedge \bar{\omega}_{PQ} &= \int_{\partial \mathcal{M}'} f \bar{\omega}_{PQ} = \sum_{i=1}^g \int_{a_i + a_i^{-1}} f \bar{\omega}_{PQ} + \sum_{j=1}^g \int_{b_j + b_j^{-1}} f \bar{\omega}_{PQ} \\ &\quad + \int_{\gamma_Q + \gamma_Q^{-1}} f \bar{\omega}_{PQ} + \oint_{c_Q} f \bar{\omega}_{PQ} + \int_{\gamma_P + \gamma_P^{-1}} f \bar{\omega}_{PQ} + \oint_{c_P} f \bar{\omega}_{PQ}. \\ &= \sum_{i=1}^g [\oint_{a_i} \omega \oint_{b_i} \bar{\omega}_{PQ} - \oint_{b_i} \omega \oint_{a_i} \bar{\omega}_{PQ}] + \oint_{c_P} \omega \int_Q^P \bar{\omega}_{PQ} - \int_Q^P \omega \oint_{c_P} \bar{\omega}_{PQ} \end{aligned} \quad (\text{C.18})$$

This is exactly the result which one would expect for the corresponding degenerate compact Riemann surface of genus  $g + 1$ . The closed path  $c_P$  corresponds to the additional  $a$ -cycle and the path from  $P$  to  $Q$  gives the  $b$ -cycle.

The more general case with two meromorphic differentials may be treated in an analogous fashion,

$$\begin{aligned} \int_{\mathcal{M}'} \omega_{P_1 P_2} \wedge \bar{\omega}_{P_3 P_4} &= \sum_{i=1}^g [\oint_{a_i} \omega_{P_1 P_2} \oint_{b_i} \bar{\omega}_{P_3 P_4} - \oint_{b_i} \omega_{P_1 P_2} \oint_{a_i} \bar{\omega}_{P_3 P_4}] \\ &\quad + \oint_{c_{P_1}} \omega_{P_1 P_2} \int_{P_2}^{P_1} \bar{\omega}_{P_3 P_4} - \int_{P_2}^{P_1} \omega_{P_1 P_2} \oint_{c_{P_1}} \bar{\omega}_{P_3 P_4} \\ &\quad + \oint_{c_{P_3}} \omega_{P_1 P_2} \int_{P_4}^{P_3} \bar{\omega}_{P_3 P_4} - \int_{P_4}^{P_3} \omega_{P_1 P_2} \oint_{c_{P_3}} \bar{\omega}_{P_3 P_4} \end{aligned}$$

$$= \sum_{i=1}^{g+3} [\oint_{a_i} \omega_{P_1 P_2} \oint_{b_i} \bar{\omega}_{P_3 P_4} - \oint_{b_i} \omega_{P_1 P_2} \oint_{a_i} \bar{\omega}_{P_3 P_4}]. \quad (\text{C.19})$$

This expression becomes logarithmically divergent as soon as two points coincide, as is clear from the l.h.s. already.

For a  $p$ -punctured surfaces and extended cohomology basis  $\omega_i$  for  $i = 1, \dots, g + p - 1$  we then get

$$\int_{\mathcal{M}'} \omega_i \wedge \bar{\omega}_j = \sum_{i=1}^{g+p-1} [\oint_{a_i} \omega_i \oint_{b_i} \bar{\omega}_j - \oint_{b_i} \omega_i \oint_{a_i} \bar{\omega}_j]. \quad (\text{C.20})$$

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